Exercise 1: [Finite volumes]

Consider a single-phase flow in the porous medium domain \( \Omega = \Omega \cup \partial \Omega = [0, 1] \times [0, 1] \):

\[
S_s \frac{\partial p(x,t)}{\partial t} - \nabla \cdot \left( \frac{K}{\mu} \nabla p(x,t) \right) = f(x,t), \quad (x,t) \in \Omega \times (0,T],
\]

\[
p(x,t) = p_D, \quad x \in \Gamma_D, \quad -K \nabla p(x,t) \cdot n = v_N, \quad x \in \Gamma_N, \quad t \in (0,T]
\]

where \( S_s \) is the mass storativity coefficient, \( p \) is the fluid pressure, \( K \) is the intrinsic permeability tensor, \( \mu \) is the fluid viscosity, and \( n \) is the unit outward normal vector at the boundary \( \partial \Omega = \Gamma_D \cup \Gamma_N \) with \( \Gamma_D \cap \Gamma_N = \emptyset \).

a) Develop an implicit second-order in space cell-centered finite volume scheme for problem (1)-(3).

b) Construct the matrix \( A \) and the right-hand side \( b \) of the discrete system of equations \( A_p^{n+1} = b \) for the grid steps \( h_x = 0.25, h_y = 0.5, \Delta t = 0.01 \), the parameters \( S_s = 0.01 \) and \( \mu = 0.01 \), the source term \( f(x,y) = x + y \), the Dirichlet and Neumann boundary conditions \( p_D(x,y) = 1 \), and \( v_N(x,y) = 0 \). The permeability tensor \( K(x,y) \) is isotropic and depicted below.

![Permeability Tensor Diagram]

\[
K_1 = \begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^{-4} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Exercise 2: [Nonlinear equations]

In the domain \( \Omega \), consider the following nonlinear elliptic equation

\[
- \nabla \cdot \left( K(p) \nabla p \right) = f(x), \quad x \in \Omega.
\]

Develop a finite-volume scheme for the case \( K(p) = \text{diag}(p^2, p) \). How to solve this problem?