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Elliptic Partial Differential Equations with Random Jump Diffusion Coefficients

MLMC-FEM Simulation

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Motivation

The modeling of stationary subsurface and fractured porous media flows by elliptic equations, where:

- A random diffusion coefficient accounts for uncertain permeability and insufficient measurements.
- Random discontinuities in the diffusion coefficient are incorporated to model heterogeneous media and fractures in ground layers.

Elliptic Problem with Jump Diffusion Coefficient

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{D} \subset \mathbb{R}^d$ a bounded, connected Lipschitz domain for some $d \in \mathbb{N}$. We consider the random elliptic problem

$$\begin{aligned} -\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) &= f(\omega, x) \quad \text{in } \Omega \times \mathcal{D}, \\ u(\omega, x) &= 0 \quad \text{on } \Omega \times \Gamma_1, \\ a(\omega, x)\vec{n} \cdot \nabla u(\omega, x) &= g(\omega, x) \quad \text{on } \Omega \times \Gamma_2, \end{aligned}$$

where

- $a : \Omega \times \mathcal{D} \to \mathbb{R}$ is a stochastic jump diffusion coefficient,
- $f : \Omega \times \mathcal{D} \to \mathbb{R}$ is a random source function,
- $\partial\mathcal{D} = \Gamma_1 \dot\cup \Gamma_2$ with $|\Gamma_1| > 0$ and Γ_2 such that the exterior normal derivative $\vec{n} \cdot \nabla u$ is well-defined for any $u \in C^1(\overline{\mathcal{D}})$ and
- $g : \Omega \times \Gamma_2 \to \mathbb{R}$ is the Neumann part of the boundary conditions.

Tthe diffusion coefficient takes the following shape:

$$a(\omega, x) := \overline{a}(x) + \Phi(W(\omega, x)) + P(\omega, x),$$

where $H := L^2(\mathcal{D})$ and

- $\overline{a}, \Phi \in C^1(\mathcal{D}; \mathbb{R}_{>0})$ (i.e. $\Phi(w) = \exp(w)$).
- W is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class operator $Q : H \to H$.
- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D})$, $\omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$ is a random partition of $\mathcal{D}$, where the number τ of elements in $\mathcal{T}$ is a $\mathbb{N}$-valued random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.
- $(P_i)_{i\in\mathbb{N}}$ is a sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with arbitrary non-negative distribution(s) and

$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} \mathbf{1}_{\{\mathcal{T}_i\}}(x) P_i(\omega).$$

The sequence $(P_i)_{i\in\mathbb{N}}$ is independent of τ (but not necessarily i.i.d.).

$\Longrightarrow$ Under natural assumptions on a, f and g, it can be shown that there exists $\mathbb{P}$-a.s. a unique weak solution $u \in L^2(\Omega; H^1(\mathcal{D}))$.
$\Longrightarrow$ In general, this solution is not available in closed form and it is not possible to draw unbiased samples from u.

Approximation of *a* and Finite Element Method

In order to draw approximate samples of u, it is usually necessary to approximate the diffusion coefficient a. First, the Gaussian random field W is replaced by W_N, where $N \in \mathbb{N}$ represents the cutoff-index of the Karhúnen-Loève expansion of W. In addition, only biased samples $\widetilde{P}_i$ of the jump heights P_i might be generated such that

$$||\widetilde{P}_i - P_i||_{L^2(\Omega;\mathbb{R})} \leq \sqrt{\epsilon}, \quad \text{for some } \epsilon > 0.$$

For example, if P_i follows a non-standard distribution, this error may stem from *Fourier Inversion sampling*, see [?]. This yields an approximation $a_{N,\epsilon}$ of a, which is then utilized to simulate pathwise FEM solutions $u_{N,\epsilon,h}(\omega, \cdot) \approx u(\omega, \cdot)$. The parameter $h > 0$ refers to the mesh width of the corresponding FEM triangulation.

$\Longrightarrow$ Given that $u \in L^2(\Omega; H^m(\mathcal{D}))$ for some $m \in (1, 2]$,

$$||u - u_{N,\epsilon,h}||_{L^2(\Omega;H^1(\mathcal{D}))} \leq C_1 \left(\sqrt{\sum_{i>N} \eta_i} + \sqrt{\epsilon} + h^{m-1} \right),$$

where $(\eta_i)_{i\in\mathbb{N}}$ are the eigenvalues of Q and $C_1 > 0$ a constant.
$\Longrightarrow$ To increase the order of convergence with respect to h in the finite dimensional approximation, the FEM triangulation should be chosen pathwise accordingly to the sampled diffusion coefficient, see Fig. 1 and 2.

Examples of 2D Diffusion Coefficients

The specific structure of a allows for a very flexible modeling of the random diffusion coefficient. Changes in permeability may be captured by the random partition $\mathcal{T}$ and distribution of the jump heights P_i. Uncertain diffusivities within the partition elements are represented by the continuous Gaussian part $\Phi(W)$. This allows, for instance, the modeling of *fractured media* as in Fig. 1 or *media with inclusions*, see Fig. 2.

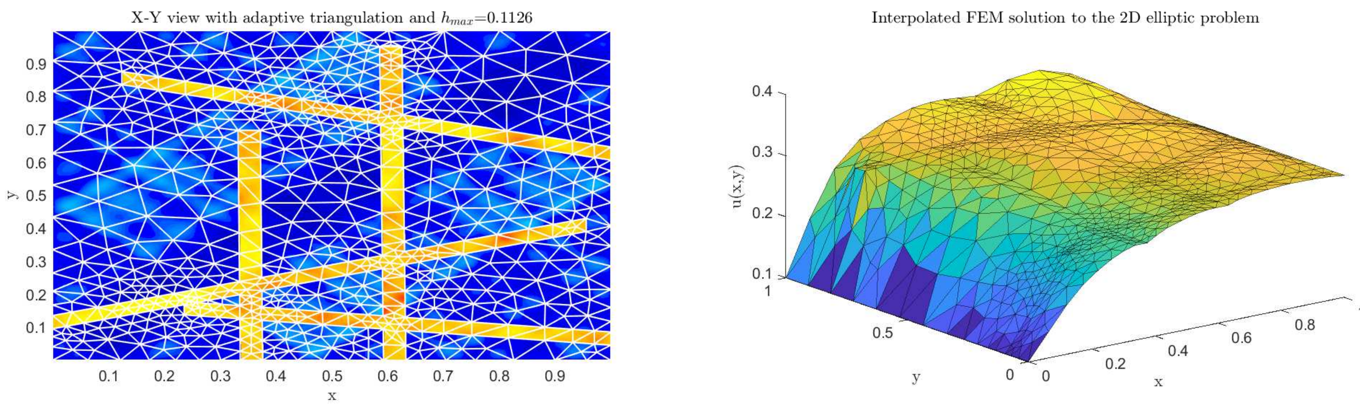

Figure 1: Sample of a medium with fractures and plot of the sampled FEM solution.

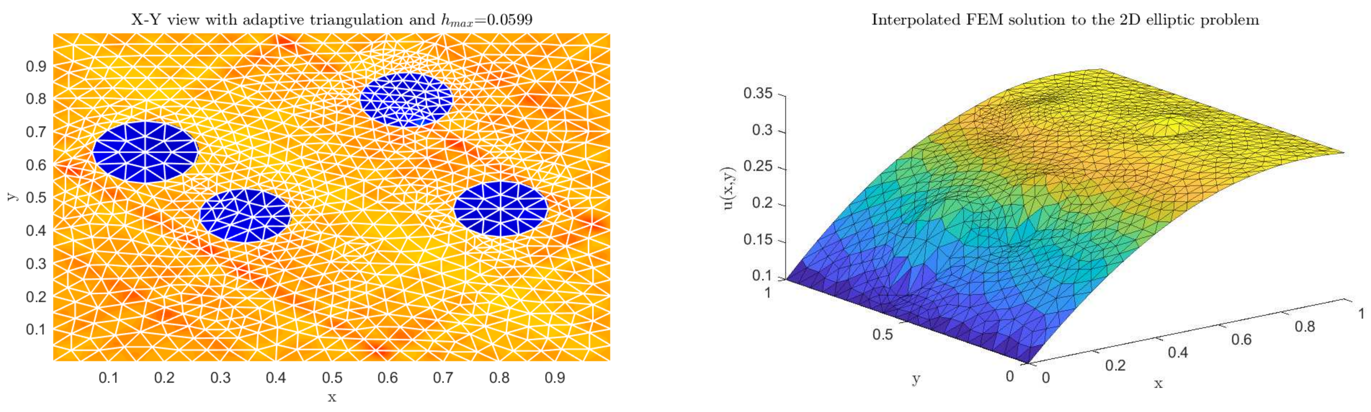

Figure 2: Sample of a medium with inclusions and plot of the sampled FEM solution.

Multilevel Monte Carlo Moment Estimation

The moments of u (expected value, variance etc.) are estimated using the *Multilevel Monte Carlo* (MLMC) method: Let $L \in \mathbb{N}$ and consider the sequences of approximation parameters $h_0 > \dots > h_L$, $\epsilon_0 > \dots > \epsilon_L$ and $N_0 < \dots < N_L$. The MLMC estimator of $\mathbb{E}(u)$ is then defined as

$$E^L(u_{N_L,\epsilon_L,h_L}) := \sum_{l=0}^{L} \frac{1}{M_l} \sum_{i=1}^{M_l} u^{(i)}_{N_l,\epsilon_l,h_l} - u^{(i)}_{N_{l-1},\epsilon_{l-1},h_{l-1}},$$

where $M_0 > \dots > M_L$ are the decreasing numbers of sampled differences and $u^{(i)}_{N_l,\epsilon_l,h_l} - u^{(i)}_{N_{l-1},\epsilon_{l-1},h_{l-1}}$ are generated independently in i on each level l. It is possible to adjust the truncation indices N_l, the sampling bias ϵ_l and the number of samples M_l throughout the levels to obtain an overall error of

$$||\mathbb{E}(u) - E^L(u_{N_L,\epsilon_L,h_L})||_{L^2(\Omega;H^1(\mathcal{D}))} \leq C_2 h_L^{m-1},$$

where $C_2 > 0$ is independent of N_L, ϵ_L, h_L and M_L.

Numerical Results

$\Longrightarrow$ Combining the MLMC estimator with an adaptive pathwise triangulation leads to faster convergence to the expected value $\mathbb{E}(u)$ and produces a lower error for any computational budget, see Fig. 3.
$\Longrightarrow$ The non-adaptive algorithm can be further enhanced by *bootstrapping* (BS), meaning the simulated quantities $u^{(i)}_{N_l,\epsilon_l,h_l}$ are "recycled" on the next level $l + 1$.

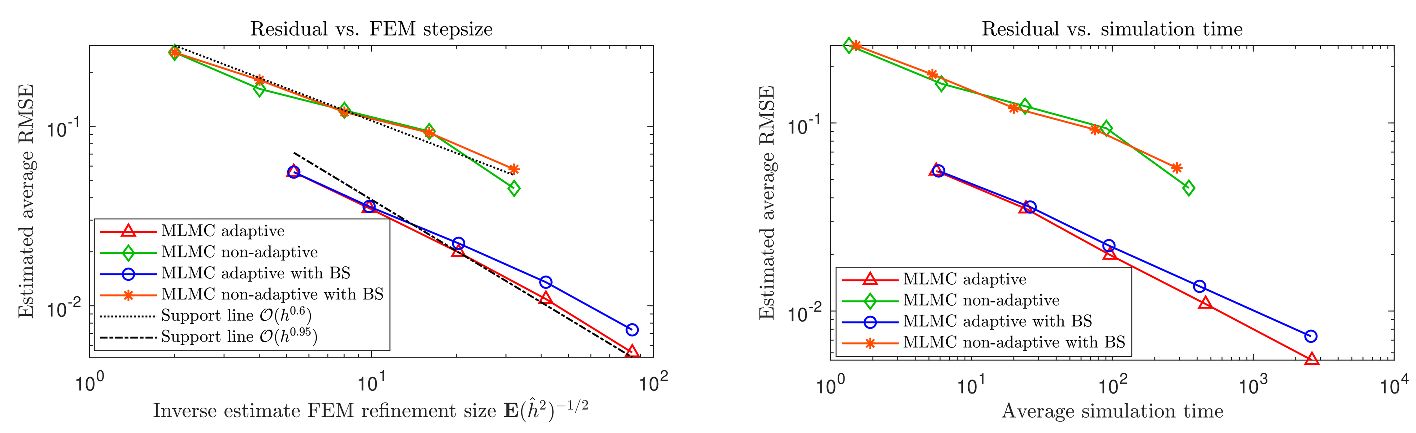

Figure 3: Numerical results for the example of a medium with fractures. Left: Convergence of the root-mean-squared error. Right: Time-to-error plot.

References

[1] A. Barth and A. Stein.
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[2] A. Barth and A. Stein.
A study of elliptic partial differential equations with jump diffusion coefficients.
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