Partial Differential Equations with Random Discontinuous Coefficients

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Motivation

The modeling of stationary/time-dependent subsurface and fractured porous media flows by elliptic/parabolic equations, where:

- Random advection- and diffusion coefficients account for uncertain permeability and insufficient measurements.
- Random discontinuities in the coefficients are incorporated to model heterogeneous media and fractures in ground layers.

Advection-Diffusion Problem with Jump Coefficients

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space, $\mathcal{D} \subset \mathbb{R}^d$ be a bounded and convex spatial domain for some $d \in \mathbb{N}$ and $\mathbb{T} = [0, T]$ with $T > 0$ be a finite time interval. We consider the random parabolic problem to find $u : \Omega \times \mathcal{D} \times \mathbb{T} \to \mathbb{R}$ such that

$$\partial_t u(\omega, x, t) + A(\omega, x)u(\omega, x, t) = f(\omega, x, t), \quad \text{in } \Omega \times \mathcal{D} \times \mathbb{T},$$

subject suitable boundary conditions on $\partial\mathcal{D}$ and initial data u_0. In the above equation, $A(\omega, x) : H^1(\mathcal{D}) \to H^{-1}(\mathcal{D})$ is a linear, elliptic differential operator of second order and f is a given source term.
To model uncertain permeability and fractured media, the advection and diffusion coefficients in $A := \nabla \cdot (a\nabla) + b\nabla\cdot$ admit the structure

$$a(\omega, x), b(\omega, x) \simeq \exp(W(\omega, x)) + P(\omega, x),$$

where

- $W : \Omega \times \mathcal{D} \to \mathbb{R}$ is a continuous Gaussian random field associated to a non-negative, symmetric trace class operator and
- $P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}$ is a discontinuous jump random field which partitions the domain $\mathcal{D}$ into non-empty subsets.

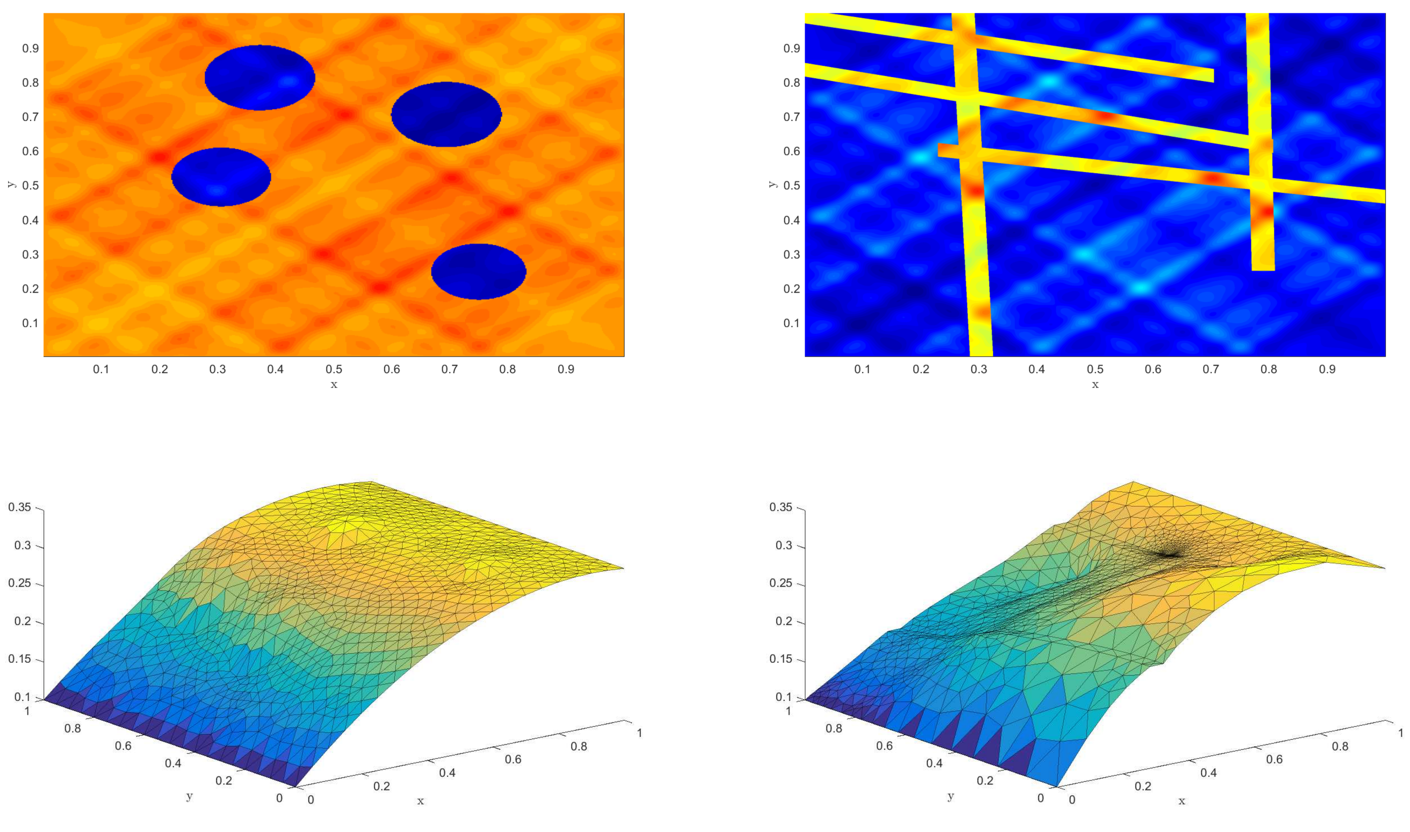

Figure 1: Samples of different diffusion coefficients (top) and corresponding pathwise solutions u (bottom).

Adaptive Numerical Schemes

In general, for fixed $\omega \in \Omega$, the exact pathwise solution $u(\omega)$ for a given sample $A(\omega, \cdot)$ will be out of reach. To obtain approximate samples of u, we need to employ several numerical techniques:

- The advection and diffusion coefficients have to be replaced by analytically tractable random fields, i.e. with truncated KL expansions, circulant embedding methods and Fourier inversion.
- A spatial discretization of $H^1(\mathcal{D})$ using finite dimensional subspaces is necessary, for instance by the Finite Element (FE) method.
- If the problem is time-dependent, we need to employ a stable time-stepping scheme.

Especially the spatial discretization involves difficulties since $u(\omega)$ has low regularity due to the jumps in the random field P. To this end, we align the FE mesh pathwise for each ω to the discontinuities in the sample $P(\omega, \cdot)$. This pathwise adaptive FE approach is then combined with suitable approximations of a, b and an implicit time stepping scheme to approximate the samples $u(\omega)$ with increased order of convergence.

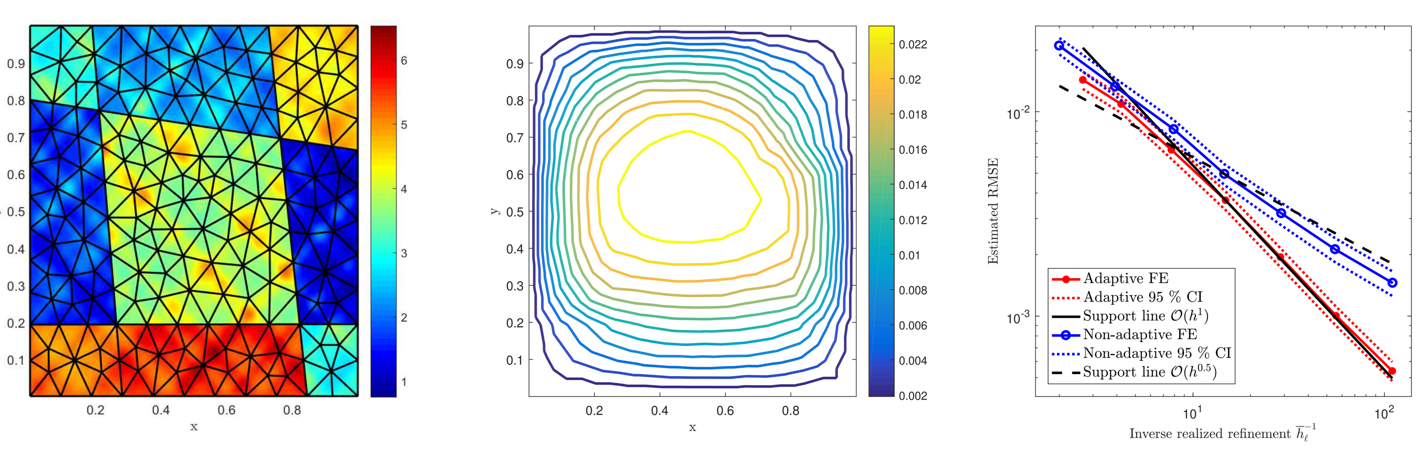

Figure 2: Sample of a diffusion/advection coefficient with adaptive FE grid (right), contour of the FE approximation to a corresponding parabolic problem (center) and convergence rates in the $L^2(\mathbb{T}; H^1(\mathcal{D}))$-norm (left). The error curves are estimated based on 100 independent samples.

Multilevel Monte Carlo Moment Estimation

For practical applications, the aim is often to estimate moments of u rather than only generating pathwise approximations. Based on each sample of A, the adaptive FE approach yields a strictly decreasing sequence of stochastic refinement parameters $(h_\ell(\omega), \ell \in \mathbb{N}_0) \subset \mathbb{R}_{>0}$. If $u_\ell(\omega)$ denotes the semi-discrete FE approximation of $u(\omega)$, it holds that

$$||u(\omega) - u_\ell(\omega)|| \leq C_1 h_\ell(\omega)^\alpha \leq C_1 \overline{h}_\ell^\alpha, \quad \ell \in \mathbb{N}_0.$$

Above, the constants $C_1, \alpha, \overline{h}_\ell$ are independent of ω, $h_\ell(\omega) \leq \overline{h}_\ell$ holds almost surely and $\overline{h}_\ell \to 0$ as $\ell \to \infty$. The other error contributions, i.e. the approximation of a, b and the temporal discretization, are now equilibrated to the spatial approximation error of order $\overline{h}_\ell$ and we denote by $\overline{u}_\ell$ the full approximation of u for each $\ell \in \mathbb{N}_0$. We then fix a (maximum) level $L \in \mathbb{N}$ and estimate $\mathbb{E}(u)$ by the *multilevel Monte Carlo estimator*

$$E^L(u_L) := \sum_{l=0}^{L} \frac{1}{M_l} \sum_{i=1}^{M_l} \overline{u}_\ell^{(i)} - \overline{u}_{\ell-1}^{(i)},$$

of $\mathbb{E}(\overline{u}_L)$, where $M_0 > \dots > M_L$ are the decreasing numbers of sampled differences and $\overline{u}_\ell^{(i)} - \overline{u}_\ell^{(i)}$ are generated independently in i on each level ℓ. The number of samples on each level may now be chosen such that the mean-squared estimation error is bounded by

$$\mathbb{E}\Big(\int_0^T ||\mathbb{E}(u) - E^L(\overline{u}_L)||^2 dt\Big)^{1/2} \leq C_2 h_L^\alpha,$$

where $C_2 > 0$ is independent of h_L and α.

$\Longrightarrow$ Combining the MLMC estimator with adaptive pathwise schemes leads to increased convergence rates and produces a lower error for any computational budget, see Figure 3 and Figure 4.
$\Longrightarrow$ The algorithm can be further enhanced by *bootstrapping* (BS), meaning the simulated quantities $u_\ell^{(i)}$ are "recycled" on the next level $\ell + 1$.

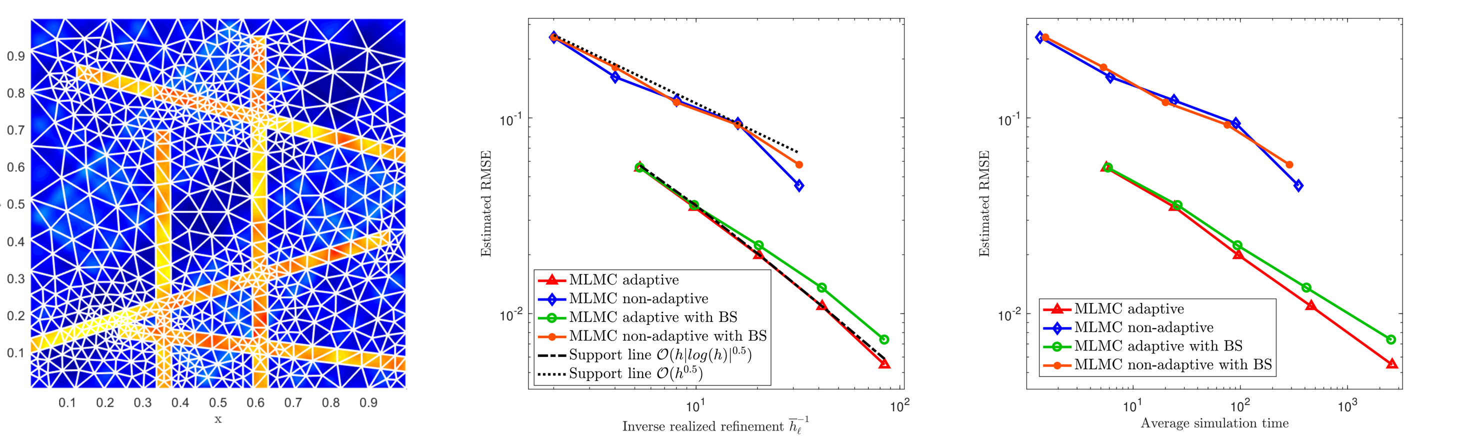

Figure 3: Coefficient for a fractured porous medium and adaptive FE grid(left), convergence plot in the $H^1(\mathcal{D})$-norm (center), Time-to-error plot (right).

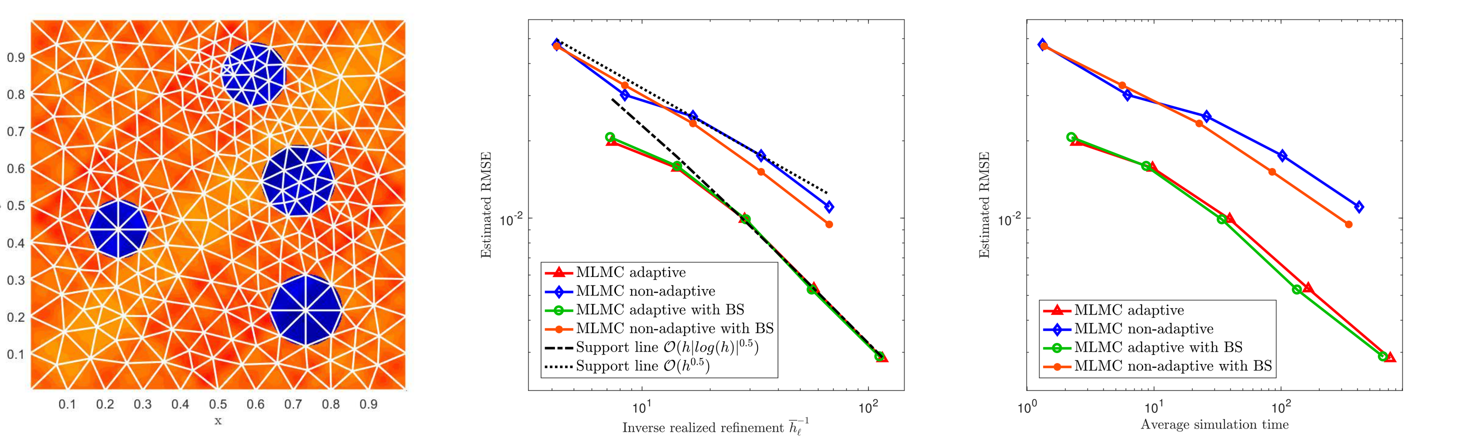

Figure 4: Coefficient for a medium with inclusions and adaptive FE grid(left), convergence plot in the $H^1(\mathcal{D})$-norm (center), Time-to-error plot (right).

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