Motivation: Random partial differential equations

Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a Hilbert space \((H, \langle \cdot, \cdot \rangle_H)\), we consider the random partial differential equation

\[
\frac{\partial}{\partial t}(t, x)u(t, x) + F(t, x)u(t, x). \quad (t, x) \in T \times D
\]
on a time interval \(T = [0, T]\) and spatial domain \(D \subset \mathbb{R}^d\) equipped with some initial and boundary conditions and

- a stochastic operator \(A\) acting on \(H\), for example \(A = \nabla \cdot (L(t, x)\nabla)\), where \(L = \{L(t, \cdot), t \in T\}\) is a \(H\)-valued stochastic process
- a (possibly nonlinear) operator \(F : H \to H\).

This equation could be, for instance, a stochastic version of the porous media equation. For various applications it might be more realistic to model \(L\) discontinuously, i.e. as an infinite-dimensional Lévy process, then also called Lévy field.

Approximation and simulation of Lévy fields

Truncated Karhunen-Loève expansions and GH fields

Let \(L = \{L(t), t \in T\}\) be a square-integrable, \(H\)-valued Lévy process on \(T\). Then there exists a non-negative, symmetric trace class covariance operator \(Q\) on \(H\) with a sequence of orthonormal eigenpairs \((\varphi_i, e_i), i \in \mathbb{N}\) and \(L\) admits the spectral decomposition

\[
L(t) = \sum_{i \in \mathbb{N}} \langle L(t), \varphi_i \rangle e_i \varphi_i.
\]

The Lévy field \(L\) may then be approximated by the truncated sum

\[
L_N(t) = \sum_{i = 0}^{N} \sqrt{e_i} \varphi_i(t),
\]

where \(\{\varphi_i, i \in \mathbb{N}\}\) is a sequence of one-dimensional Lévy processes on \(T\). An important subclass of Lévy fields with various applications in finance and physics are generalized hyperbolic (GH) fields, which are based on the generalized hyperbolic distribution. In this case, the marginal processes \(\varphi_i\) are given by one-dimensional GH processes. We use the representation of GH processes as subordinated Brownian Motions to obtain

\[
L_N(t) = \sum_{i = 0}^{N} \sqrt{e_i} \varphi_i \left( it + \beta Y(t) + \sqrt{\rho} \cdot \mathbf{W}_N(Y(t)) \right),
\]

where \(\mu, \beta \in \mathbb{R}^N, \Gamma \in \mathbb{R}^{N \times N}\), are parameters of the GH field \(L\), \(\mathbf{W}_N\) is a \(N\)-dimensional Brownian motion and \(Y = \{Y(t), t \in T\}\) a generalized inverse Gaussian process independent of \(\mathbf{W}_N\). Formula (1) yields an efficient sampling algorithm for \(L_N\), whose performance is independent of the truncation index \(N\). Further, we have constructed \(L_N\) in such a way, that the approximation itself is a Lévy field with known marginal distributions. In other words, we know the law of the process \((L_N(t, x), t \in T)\) for an arbitrary fixed \(x \in D\).

Often, as in the GH case, the processes \(\varphi_i\) have infinite activity, i.e. infinitely many jumps in every compact time interval for \(t\)-almost all trajectories, and, therefore, need to be approximated.

Fourier inversion method for one-dimensional processes

For a broad class of one-dimensional Lévy processes \(\varphi_i\), the characteristic function \(\varphi_i : \mathbb{R} \to \mathbb{C}\) is explicitly available. We exploit the knowledge of \(\varphi_i\) and a Fourier inversion property of characteristic functions to draw samples of the increment \(\varphi_i(t + \Delta t) - \varphi_i(t)\), where \(\Delta t > 0\) is a small step in time. This procedure results in a piecewise constant approximation \(\tilde{\varphi_i}\) of \(\varphi_i\) on \(T\) and involves numerical integration and therefore a certain error. With some relatively weak assumptions on \(\varphi_i\), we have shown that \(\tilde{\varphi_i}\) converges to \(\varphi_i\) in distribution. Further, if \(\Delta t\) has a finite \(p - \text{th}\) moment for \(p \geq 1\), then we have the error bound

\[
\sup_{t \in T} |\varphi_i(t) - \tilde{\varphi_i}(t)| < C \Delta t^{1/p}
\]

with some constant \(C > 0\), hence \(\tilde{\varphi_i}\) converges in \(L^p([0, T])\) towards \(\varphi_i\) uniformly on \(T\). Since \(\varphi_i\) can be determined or at least bounded from above, this new result allows us to quantify the approximation error of infinite-dimensional Lévy processes.

Simulation of Lévy fields and mean-square error estimation

Including the aforementioned discretization, we are able to approximate a given \(H\)-valued Lévy process \(L\) by

\[
L_N(t) = \sum_{i \in \mathbb{N}} \sqrt{e_i} \tilde{\varphi_i}(t),
\]

where the processes \(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N\) are obtained by Fourier inversion. The mean-square approximation error in \(H\) is then bounded by

\[
\sup_{t \in T} |L(t) - L_N(t)|_{L^2([0,T],H)} \leq C_2 \Delta t^{1/2} \left( \sum_{i \in \mathbb{N}} \rho_i \right)^{1/2} + \left( \sum_{i \in \mathbb{N}} \rho_i \right)^{1/2}
\]

and we achieve \(\tilde{L}_N(x,t) \to L(x,t)\) on \(T\) as \(N \to \infty\) and \(\Delta t \to 0\).

Numerical examples

As a test for the novel approximation method, we simulate hyperbolic Lévy fields, which is next to the normal inverse Gaussian (NIG) field the most popular example of a GH Lévy field. In the simulations below, \(T := [0, T]\), \(\Delta t := 2^{-6}\) and the spatial domain is \(D := [0, 1]\). We use two different covariance operators to show the effects of a varying roughness of \(Q\) on the generated fields. To equilibrate both contributions in \(\rho_0\), \(N\) is coupled to \(\Delta t\) and the decay of the eigenvalues \(\rho_i\) of \(Q\). As Table 1 indicates, the truncation index decreases as the correlation throughout the field increases.

Table 1 Approximation errors and simulation times for hyperbolic Lévy fields.

<table>
<thead>
<tr>
<th>Truncation index</th>
<th>L^2(H) error in % of the overall variance</th>
<th>Simulation time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential covariance</td>
<td>132</td>
<td>6.30%</td>
</tr>
<tr>
<td>Matérn-1.5 covariance</td>
<td>18</td>
<td>6.07%</td>
</tr>
</tbody>
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References

