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Computational Methods for Uncertainty Quantification
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Infinite-Dimensional Lévy Processes

Approximation and Simulation

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Motivation: Random partial differential equations

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Hilbert space $(H, (\cdot,\cdot)_H)$, we consider the random partial differential equation

$$\frac{\partial}{\partial t}u(t.x) = A(t,x)u(t.x) + F(u(t,x)), \qquad (t,x) \in \mathbb{T} \times \mathcal{D}$$

on a time interval $\mathbb{T} = [0,T]$ and spatial domain $\mathcal{D} \subset \mathbb{R}^n$ equipped with some initial and boundary conditions and

- a stochastic operator A acting on H, for example $A = \nabla \cdot (L(t,x)\nabla)$, where $L = (L(t,\cdot), t \in \mathbb{T})$ is a H-valued stochastic process
- a (possibly nonlinear) operator $F : H \to H$.

This equation could be, for instance, a stochastic version of the porous media equation. For various applications it might be more realistic to model L discontinuously, i.e. as an infinite-dimensional *Lévy process*, then also called *Lévy field*.

Approximation and simulation of Lévy fields

Truncated Karhunen-Loève expansions and GH fields

Let $L = (L(t), t \in \mathbb{T})$ be a square-integrable, H-valued Lévy process on $\mathbb{T}$. Then there exists a non-negative, symmetric trace class covariance operator Q on H with a sequence of orthonormal eigenpairs $((\rho_i, e_i), i \in \mathbb{N})$ and L admits the spectral decomposition

$$L(t) = \sum_{i\in\mathbb{N}} (L(t), e_i)_H e_i.$$

The Lévy field L may then be approximated by the truncated sum

$$L_N(t) = \sum_{i\in\mathbb{N}} \sqrt{\rho_i} e_i \ell_i(t),$$

where $(\ell_i, i \in \mathbb{N})$ is a sequence of one-dimensional Lévy processes on $\mathbb{T}$. An important subclass of Lévy fields with various applications in finance and physics are *generalized hyperbolic (GH) fields*, which are based on the generalized hyperbolic distribution. In this case, the marginal processes ℓ_i in L resp. L_N are given by one-dimensional GH processes. We use the representation of GH processes as subordinated Brownian Motions to obtain

$$L_N(t) = \sum_{i=1}^{N} \sqrt{\rho_i} e_i \left(\mu t + \Gamma\beta Y(t) + \sqrt{\Gamma} W_N(Y(t))\right)_i, \tag{1}$$

where $\mu, \beta \in \mathbb{R}^N$, $\Gamma \in \mathbb{R}^{N\times N}$ are parameters of the GH field L, W_N is a N-dimensional Brownian motion and $Y = (Y(t), t \in \mathbb{T})$ a *generalized inverse Gaussian process* independent of W_N. Formula (1) yields an efficient sampling algorithm for L_N whose performance is independent of the truncation index N. Further, we have constructed L_N in such a way, that the approximation itself is a Lévy field with known marginal distributions. In other words, we know the law of the process $(L_N(t,x), t \in \mathbb{T})$ for an arbitrary fixed $x \in \mathcal{D}$.
Often, as in the GH case, the processes ℓ_i have *infinite activity*, i.e. infinitely many jumps in every compact time interval for $\mathbb{P}$-almost all trajectories, and, therefore, need to be approximated.

Fourier inversion method for one-dimensional processes

For a broad class of one-dimensional Lévy processes ℓ_i, the characteristic function $\phi_{\ell_i} : \mathbb{R} \to \mathbb{C}$ is explicitly available. We exploit the knowledge of ϕ_{ℓ_i} and a Fourier inversion property of characteristic functions to draw samples of the increment $\ell_i(t+\Delta t) - \ell_i(t)$, where $\Delta t > 0$ is a small step in time. This procedure results in a piecewise constant approximation $\widetilde{\ell}_i$ of ℓ_i on $\mathbb{T}$ and involves numerical integration and therefore a certain error. With some relatively weak assumptions on ϕ_{ℓ_i}, we have shown that $\widetilde{\ell}_i$ converges to ℓ_i in distribution. Further, if ℓ_i has a finite $p-th$ moment for $p \geq 1$, then we have the error bound

$$\sup_{t\in\mathbb{T}} \mathbb{E}[|\ell(t) - \widetilde{\ell}(t)|^p] < C_{\ell,p}\Delta t$$

with some constant $C_{\ell,p} > 0$, hence $\widetilde{\ell}_i$ converges in $L^p(\Omega;\mathbb{R})$ towards ℓ_i uniformly on $\mathbb{T}$. Since $C_{\ell,p}$ can be determined or at least bounded from above, this new result allows us to quantify the approximation error of infinite-dimensional Lévy processes.

Simulation of Lévy fields and mean-square error estimation

Including the aforementioned discretization, we are able to approximate a given H-valued Lévy process L by

$$\widetilde{L}_N(t) = \sum_{i\in\mathbb{N}} \sqrt{\rho_i} e_i \widetilde{\ell}_i(t),$$

where the processes $\widetilde{\ell}_1, \dots, \widetilde{\ell}_N$ are obtained by Fourier inversion. The mean-square approximation error in H is then bounded by

$$\sup_{t\in\mathbb{T}} ||L(t) - \widetilde{L}_N(t)||_{L^2(\Omega;H)} < \left(C_{\ell,2}\Delta t \sum_{i=1}^{N} \rho_i\right)^{1/2} + \left(T \sum_{i=N+1}^{\infty} \rho_i\right)^{1/2} \tag{2}$$

and we achieve $\widetilde{L}_N \xrightarrow{L^2(\Omega;H)} L$ on $\mathbb{T}$ as $N \to \infty$ and $\Delta t \to 0$.

Numerical examples

As a test for the novel approximation method, we simulate *hyperbolic Lévy fields*, which is next to the *normal inverse Gaussian (NIG)* field the most popular example of a GH Lévy field. In the simulations below, $\mathbb{T} := [0,1]$, $\Delta t := 2^{-6}$ and the spatial domain is $\mathcal{D} := [0,1]$. We use two different covariance operators to show the effects of a varying roughness of Q on the generated fields. To equilibrate both contributions in (2), N is coupled to Δt and the decay of the eigenvalues ρ_i of Q. As Table 1 indicates, the truncation index decreases as the correlation throughout the field increases.

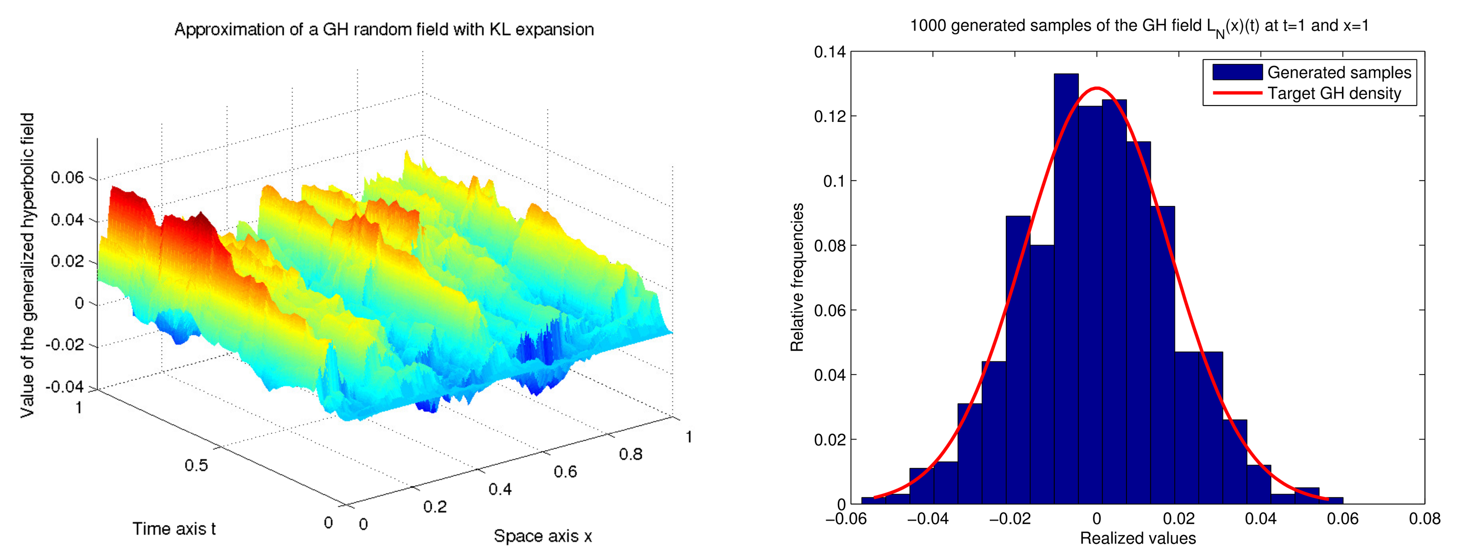

Figure 1 Sample and empirical distribution of a hyperbolic field with exponential covariance operator.

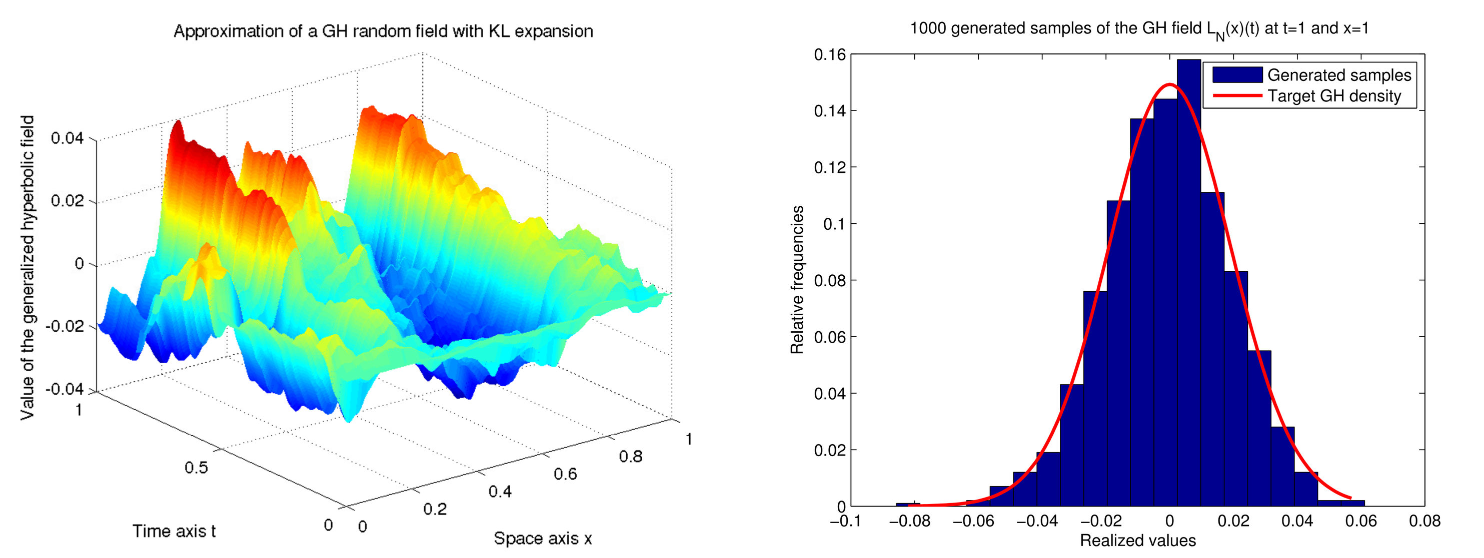

Figure 2 Sample and empirical distribution of a hyperbolic field with Matérn-1.5 covariance operator.

	Truncation index N	$L^2(\Omega;H)$-error in % of the overall variance	Simulation time
Exponential covariance	132	6.30%	0.0759 sec.
Matérn-1.5 covariance	18	6.07%	0.0752 sec.

Table 1 Approximation errors and simulation times for hyperbolic Lévy fields.

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