Motivating Example: Energy forward markets

An approach to model energy forward dynamics is to consider first order hyperbolic stochastic partial differential equations. An infinite-dimensional noise term then represents the large number of idiosyncratic risk sources in the considered markets.

Given a probability space $(\Omega, F, P)$ and a separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$, we consider the stochastic partial differential equation

$$\frac{dX(t)}{dt} = (AX(t) + a(t))dt + b(t)dL(t), \quad t \in \mathbb{T}$$

(1)

on a time interval $\mathbb{T} = [0, T]$, equipped with some boundary conditions where

- $A : \mathcal{H} \to \mathcal{H}$ is a first order differential operator
- $a : \mathbb{T} \to \mathbb{R}$ is a mapping with Bochner-integrable trajectories
- $L : \mathbb{T} \to \mathcal{H}$ is a square-integrable, $\mathcal{H}$-valued stochastic process with covariance operator $Q \in \mathcal{L}(\mathcal{H})$.
- $b : \mathbb{T} \to (\mathcal{L}(\mathcal{H})^\ast, \mathcal{H})$ is an operator-valued process.

The price of a forward contract at $t \in \mathbb{T}$ with time left to maturity $x \in \mathbb{T}$ may then be expressed by the mapping $g(X(t, x)) = \phi(t)$, for various applications it might be appropriate to model $L$ discontinuously, i.e. as an infinite-dimensional Lévy process, also called Lévy field. For any $x \in \mathbb{T}$ the marginal process $L(t, x)$ is then a one-dimensional Lévy process, meaning that $L$ should be approximated in a way that this property is preserved.

Approximation of Lévy fields via Fourier inversion

Let $L = (L(t), t \in \mathbb{T})$ be a square-integrable, $\mathcal{H}$-valued Lévy process on $\mathbb{T}$. Then there exists a non-negative, symmetric trace class covariance operator $Q$ on $\mathcal{H}$ with a sequence of orthonormal eigenpairs $(\varphi_i, \rho_i, i \in \mathbb{N})$. Further, $L$ admits the spectral decomposition

$$L(t) = \sum_{i=1}^{N} \langle L(t), \varphi_i \rangle \varphi_i$$

The Lévy field $L$ may then be approximated by the truncated sum

$$L_N(t) = \sum_{i=1}^{N} \sqrt{\rho_i} \varphi_i(t)$$

(2)

where $(\varphi_i, \rho_i, i \in \mathbb{N})$ is a sequence of one-dimensional, not necessarily independent but merely uncorrelated, Lévy processes on $\mathbb{T}$. An important subclass of Lévy fields with various applications in finance and physics are generalized hyperbolic (GH) fields, which are based on the generalized hyperbolic distribution. In this case, the marginal processes $L_i$ in $L$ resp. $L_N$ are given by one-dimensional GH processes. For this subclass, we have constructed $L_N$ in such a way, that the approximation itself is a Lévy field with known marginal distributions. In other words, we know the law of the processes $(L_N(t), t \in \mathbb{T})$ for an arbitrary fixed $x \in \mathbb{T}$.

As a test for the novel approximation method, we simulate normal inverse Gaussian (NIG) and hyperbolic Lévy fields $L$ and embed them in the discretization scheme for the SPDE (1). The PDE coefficients are chosen to be $a(t, x) = e^{-2x}x$ and $b(t, x) = e^{-2x}$ with initial condition $X_0(x) = e^{-x^2/2} \int_0^\infty c(e^{-u^2})$, and inflow boundary $X(t, \mathbb{T}) = e^{-t^2/2} \int_0^\infty c(e^{-u^2})$, where $c$ is the cumulant function of the one-dimensional NIG resp. hyperbolic Lévy process. Both the NIG and the hyperbolic field are correlated by an exponential covariance operator with kernel function $(x, y) \to e^{-|x-y|}$ on $\mathbb{T} \times \mathbb{T}$. In the simulations below, $\mathbb{T} := [0, 1]$, $\Delta t = \Delta x = 2^{-10}$ and the truncation index $N$ has been chosen to equilibrate the error terms $\sum_{j=N}^{\infty} \rho_j$ and $\sum_{i=1}^{N} \varphi_i$

Error estimation in the SPDE discretization scheme

Including the aforementioned approximation $L_N$ in the fully discrete scheme (1) is straightforward in a simulation. The overall order of convergence then depends on the Galerkin resp. Petrov-Galerkin approximation on the semidiscrete problem as well as on the time marching scheme in the fully discrete case. One may for example choose the streamline diffusion method combined with a backward Euler-Maruyama scheme to obtain a fully discrete approximation $X_{\Delta t, \Delta x}$ of the solution $X(t)$ to (1). Here, $h$ and $\Delta t$ are the refinement sizes in space resp. time and $N$ again indicates the cutoff-index in the KL-expansion (2). Assuming a certain regularity on the PDE parameters, the boundary data and the initial condition $X_0$, it is possible to show that

$$\sup_{t \in \mathbb{T}} |X_{\Delta t, \Delta x}(X_{\mathcal{L}, T}) - C_{\Delta t, \Delta x, \mathcal{L}} T \left( h^{1/2} + \sqrt{\Delta t} \left( 1 + \sum_{\rho_i} + \sum_{\rho_j} \right) \right)$$

where $\rho_i$ is the discrete grid of points in $\mathbb{T}$ with maximum distance $\Delta t$ and $C_{\Delta t, \Delta x, \mathcal{L}, T} > 0$ is an independent constant.

Numerical examples

As a test for the novel approximation method, we simulate normal inverse Gaussian (NIG) and hyperbolic Lévy fields $L$ and embed them in the discretization scheme for the SPDE (1). The PDE coefficients are chosen to be $a(t, x) = e^{-2x^2}x^2$ and $b(t, x) = e^{-2x^2}$ with initial condition $X_0(x) = e^{-x^2/2} \int_0^\infty c(e^{-u^2})$, and inflow boundary $X(t, \mathbb{T}) = e^{-t^2/2} \int_0^\infty c(e^{-u^2})$, where $c$ is the cumulant function of the one-dimensional NIG resp. hyperbolic Lévy process. Both the NIG and the hyperbolic field are correlated by an exponential covariance operator with kernel function $(x, y) \to e^{-|x-y|}$ on $\mathbb{T} \times \mathbb{T}$. In the simulations below, $\mathbb{T} := [0, 1]$, $\Delta t = \Delta x = 2^{-10}$ and the truncation index $N$ has been chosen to equilibrate the error terms $\sum_{j=N}^{\infty} \rho_j$ and $\sum_{i=1}^{N} \varphi_i$.

References


