Stochastic Partial Differential Equations with Lévy Noise

Approximation and Simulation

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Motivating Example: Energy forward markets

An approach to model energy forward dynamics is to consider first order hyperbolic stochastic partial differential equations. An infinite-dimensional noise term then represents the large number of idiosyncratic risk sources in the considered markets.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a separable Hilbert space $(H, (\cdot,\cdot)_H)$, we consider the stochastic partial differential equation

$$dX(t) = (AX(t) + a(t))dt + b(t)dL(t), \qquad t \in \mathbb{T} \tag{1}$$

on a time interval $\mathbb{T} = [0, T]$, equipped with some boundary conditions where

- $A : H \to H$ is a first order differential operator
- $a : \mathbb{T} \to H$ is a mapping with Bochner-integrable trajectories
- $L : \mathbb{T} \to H$ is a square-integrable, H-valued stochastic process with covariance operator $Q \in L_+^1(H)$.
- $b : \mathbb{T} \to L(Q^{1/2}(H), H)$ is an operator-valued process.

The price of a forward contract at $t \in \mathbb{T}$ with time left to maturity $x \in \mathbb{T}$ may then be expressed by the mapping $g(X(t,x)) = e^{X(t,x)}$. For various applications it might be appropriate to model L discontinuously, i.e. as an infinite-dimensional *Lévy process*, also called *Lévy field*. For any $x \in \mathbb{T}$ the marginal process $L(\cdot, x)$ is then a one-dimensional Lévy process, meaning that L should be approximated in a way that this property is preserved.

Approximation of Lévy fields via Fourier inversion

Let $L = (L(t), t \in \mathbb{T})$ be a square-integrable, H-valued Lévy process on $\mathbb{T}$. Then there exists a non-negative, symmetric trace class covariance operator Q on H with a sequence of orthonormal eigenpairs $((\rho_i, e_i), i \in \mathbb{N})$. Further, L admits the spectral decomposition

$$L(t) = \sum_{i \in \mathbb{N}} (L(t), e_i)_H e_i.$$

The Lévy field L may then be approximated by the truncated sum

$$L_N(t) = \sum_{i=1}^{N} \sqrt{\rho_i} e_i \ell_i(t), \tag{2}$$

where $(\ell_i, i \in \mathbb{N})$ is a sequence of one-dimensional, not necessarily independent but merely uncorrelated, Lévy processes on $\mathbb{T}$. An important subclass of Lévy fields with various applications in finance and physics are *generalized hyperbolic (GH) fields*, which are based on the generalized hyperbolic distribution. In this case, the marginal processes ℓ_i in L resp. L_N are given by one-dimensional GH processes. For this subclass, we have constructed L_N in such a way, that the approximation itself is a Lévy field with known marginal distributions. In other words, we know the law of the process $(L_N(t,x), t \in \mathbb{T})$ for an arbitrary fixed $x \in \mathbb{T}$.

For a broad class of one-dimensional Lévy processes ℓ_i, the characteristic function $\phi_{\ell_i} : \mathbb{R} \to \mathbb{C}$ is explicitly available. We exploit the knowledge of ϕ_{ℓ_i} and the Fourier inversion property of characteristic functions to draw samples of the increment $\ell_i(t + \Delta t) - \ell_i(t)$ for a small step in time $\Delta t > 0$. This procedure results in a piecewise constant approximation $\widetilde{\ell}_i$ of ℓ_i on $\mathbb{T}$ and involves numerical integration and therefore a certain error. If ℓ_i has a p-th moment and under some relatively weak assumptions on ϕ_{ℓ_i}, we have shown that $\widetilde{\ell}_i$ converges in $L^p(\Omega;\mathbb{R})$ towards ℓ_i uniformly on $\mathbb{T}$ as $\Delta t \to 0$ and derived an error estimte in the corresponding norm.

This new result allows us to quantify the approximation error of infinite-dimensional Lévy processes: For a given H-valued Lévy process L, define

$$\widetilde{L}_N(t) := \sum_{i=1}^{N} \sqrt{\rho_i} e_i \widetilde{\ell}_i(t),$$

where the processes $\widetilde{\ell}_1, \dots, \widetilde{\ell}_N$ are obtained by Fourier inversion. The mean-square approximation error in H is then bounded by

$$\sup_{t \in \mathbb{T}} \|L(t) - \widetilde{L}_N(t)\|_{L^2(\Omega;H)} \le \left(C_\ell \Delta t \sum_{i=1}^{N} \rho_i \right)^{1/2} + \left(T \sum_{i>N} \rho_i \right)^{1/2}, \tag{3}$$

where $C_\ell > 0$ is constant independent of Δt. Hence, we achieve the convergence $\widetilde{L}_N \overset{L^2(\Omega;H)}{\longrightarrow} L$ on $\mathbb{T}$ as $N \to \infty$ and $\Delta t \to 0$.

Error estimation in the SPDE discretization scheme

Including the aforementioned approximation $\widetilde{L}_N$ in the fully discrete scheme to (1) is straightforward in a simulation. The overall order of convergence then depends on the Galerkin resp. Petrov-Galerkin approximation on the semidiscrete problem as well as on the time marching scheme in the fully discrete case. One may for example choose the streamline diffusion method combined with a backward Euler-Maruyama scheme to obtain a fully discrete approximation $\widetilde{X}_{h,\Delta t,N}$ of the solution X to (1). Here, h and Δt are the refinement sizes in space resp. time and N again indicates the cutoff-index in the KL-expansion (2). Assuming a certain regularity on the PDE parameters, the boundary data and the initial condition X_0, it is possible to show that

$$\sup_{t \in \Theta_n} \|\widetilde{X}_{h,\Delta t,N} - X\|_{L^2(\Omega,H)} = C_{a,b,X_0,L,T} \left(h^{3/2} + \sqrt{\Delta t}\Big(1 + \sum_{i=1}^{N} \rho_i\Big) + \sqrt{\sum_{i>N} \rho_i} \right).$$

where Θ_n is the discrete grid of points in $\mathbb{T}$ with maximum distance Δt and $C_{a,b,X_0,L,T} > 0$ is an independent constant.

Numerical examples

As a test for the novel approximation method, we simulate *normal inverse Gaussian (NIG)* and *hyperbolic Lévy fields* L and embed them in the discretization scheme for the SPDE (1). The PDE coefficients are chosen to be $a(t,x) = e^{-2\alpha x}\sigma^2$ and $b(t,x) = e^{-\alpha x}\sigma$ with initial condition $X_0(x) = e^{-\alpha x} + \int_0^x c(e^{-\alpha s})$ and inflow boundary $X(t,T) = e^{-\alpha T} + \int_0^T c(e^{-\alpha s})$, where c is the cumulant function of the one-dimensional NIG resp. hyperbolic Lévy process. Both the NIG and the hyperbolic field are correlated by an exponential covariance operator with kernel function $(x,y) \mapsto e^{-\frac{|x-y|}{r}}$ on $\mathbb{T} \times \mathbb{T}$. In the simulations below, $\mathbb{T} := [0,1]$, $\Delta t = \Delta h = 2^{-10}$ and the truncation index N has been chosen to equilibrate the error terms $\sum_{i>N} \rho_i$ and $\Delta t(1 + \sum_{i=1}^{N} \rho_i)$.

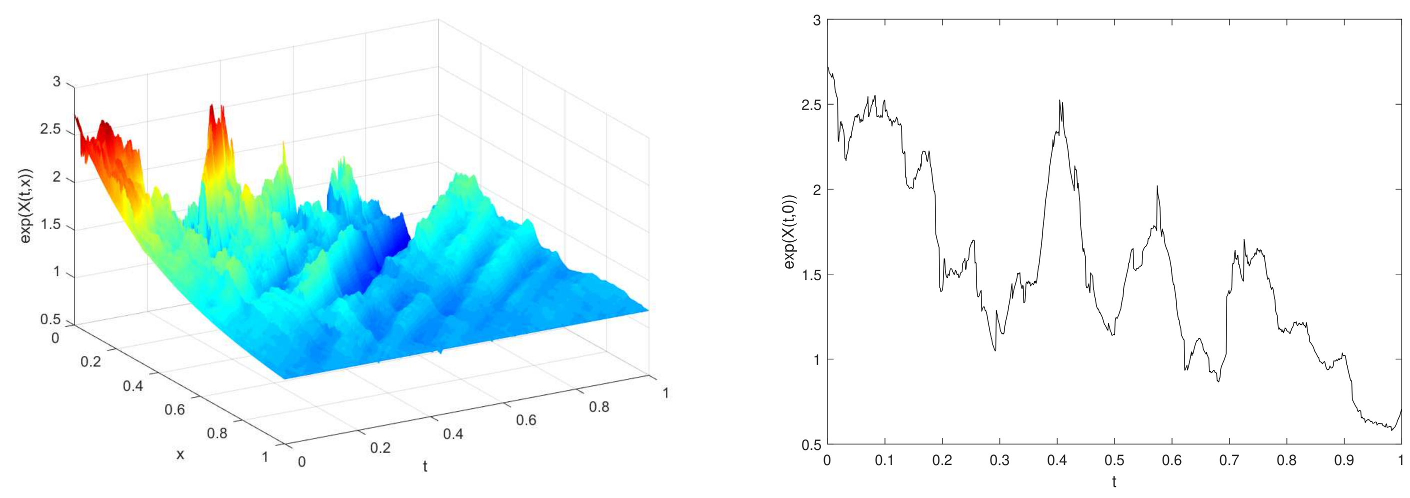

Figure 1 Forward surface and spot curve for a NIG Lévy field with $\alpha = 2$ and $\sigma = 0.5$.

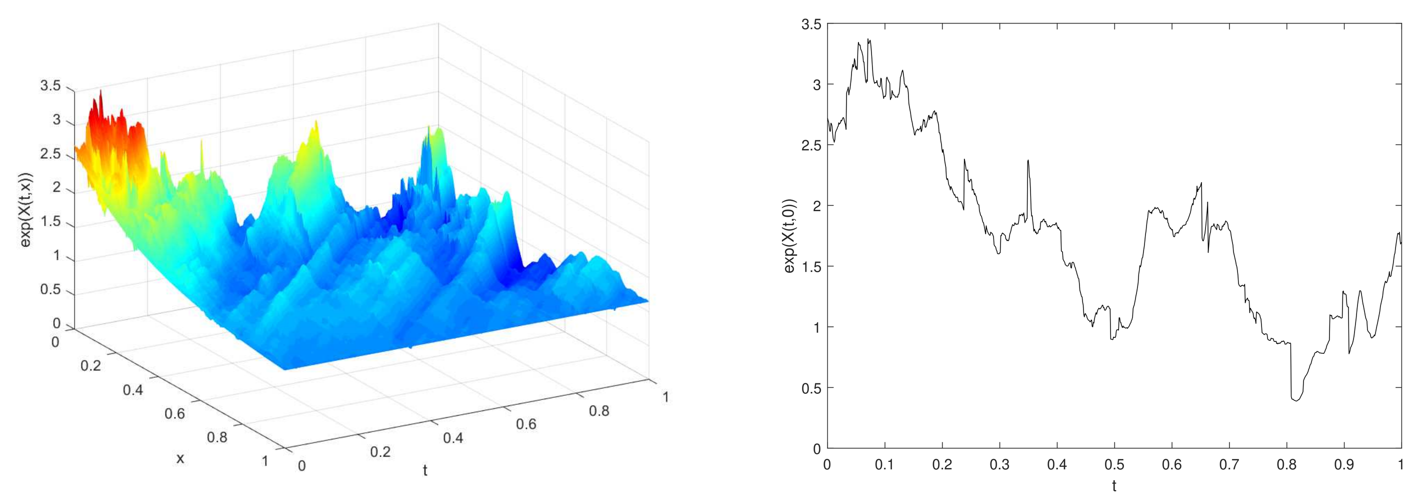

Figure 2 Forward surface and spot curve for an hyperbolic Lévy field with $\alpha = 2$ and $\sigma = 0.5$.

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