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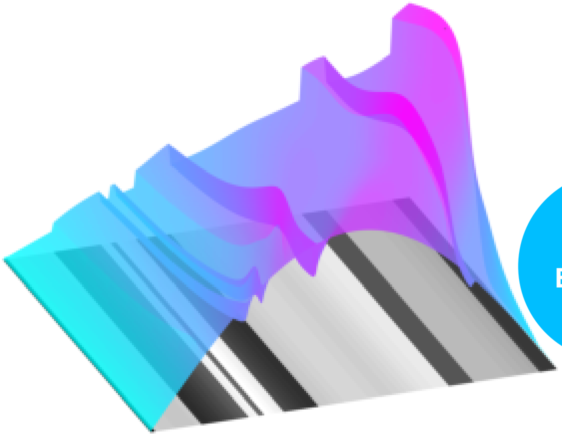

Hyperbolic conservation laws with stochastic jump coefficient

Lukas Brencher

GAMM 2020 / 2021

Joint work with Andrea Barth

March 17, 2021

ians

SimTech

Motivation & Problem description

Subsurface flow

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- insufficient measurements
- uncertain permeabilities
- fractures / heterogeneities

Stochastic Burgers' equation

$$u_t + \operatorname{div}\left(a(\omega, x)\frac{u^2}{2}\right) = 0 \qquad \text{in } \Omega \times \mathbb{X} \times \mathbb{T}$$

$$u(\omega, x, 0) = u_0(\omega, x) \qquad \text{on } \Omega \times \mathbb{X} \times \{0\}$$

- $(\Omega, \mathcal{A}, \mathbb{P})$ complete probability space
- periodic boundary conditions
- $u_0 \in L^p(\Omega; L^\infty(\mathbb{R}))$ stoch. initial condition

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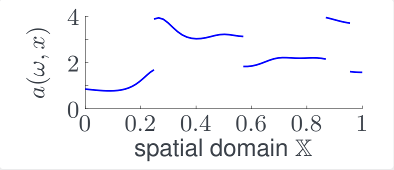
general definition to allow flexible modeling

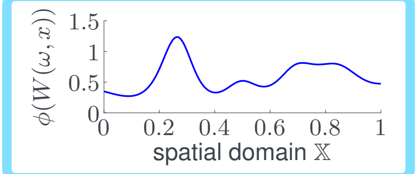
Stochastic jump coefficient

Barth and Stein (2018b)

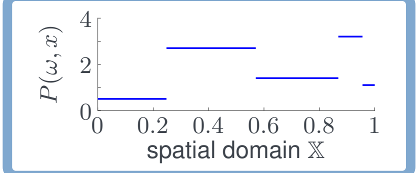
$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x)$$



Continuous part $\overline{a}(x) + \phi(W(\omega, x))$



Jump part $P(\omega, x)$



Stochastic jump coefficient

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$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x)$$

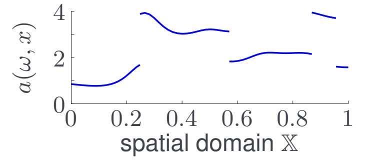
Continuous part

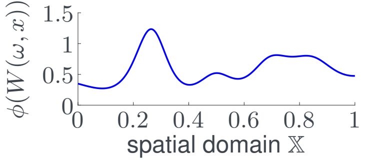
$\overline{a} \in C(\mathbb{R}; \mathbb{R}_{\geq 0})$
Deterministic mean function

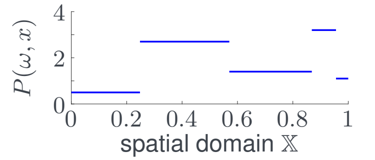
$\phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$
In our case: $\phi(\xi) = \exp(\xi)$

$W \in L^2(\Omega; L^2(\mathbb{R}))$
(Zero-mean) Gaussian random field (GRF)

Non-negative, symmetric trace class (covariance) operator $Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.







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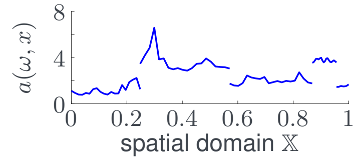
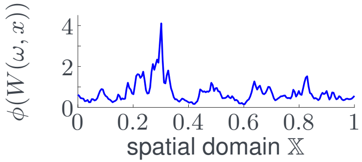
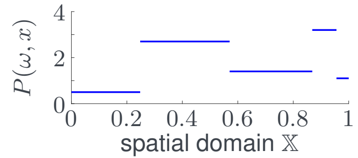






Stochastic jump coefficient

Barth and Stein (2018b)

$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x)$$

Jump part

$\mathcal{T} : \Omega \to \mathcal{B}(\mathbb{R}),\ \omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$

Random partition of $\mathbb{R}$.

$\tau : \Omega \to \mathbb{N}$

Random number of elements $\mathcal{T}_i$ in $\mathcal{T}$

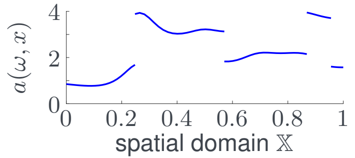
λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

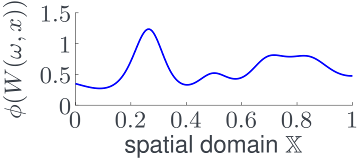
Measure controlling positions of elements $\mathcal{T}_i$

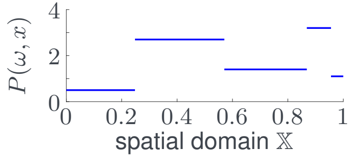
$P : \Omega \times \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau} \mathbf{1}_{\mathcal{T}_i}(x) P_i(\omega)$

$(P_i, i \in \mathbb{N})$

Random jump heights of elements $\mathcal{T}_i$







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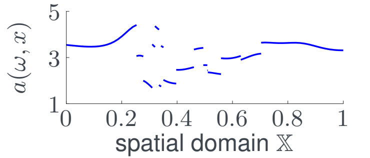
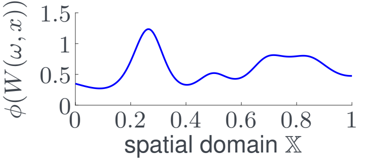
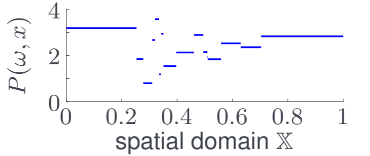






Approximation of the jump coefficient

Barth and Stein (2018a)
Gil-Pelaez (1951)
Hughett (1998)

Gaussian random field

Karhunen-Loève (KL) expansion

$$W(\omega, x) = \sum_{i=1}^{\infty} \sqrt{\eta_i} e_i(x) Z_i(\omega)$$

$Z_i \sim \mathcal{N}(0,1)$
Standard-normal random variables

$((\eta_i, e_i), i \in \mathbb{N})$
(ordered) eigenpairs of the covariance operator Q

$\Rightarrow$ **Approximation:** truncate the series after the first $N \in \mathbb{N}$ terms.

Jump field

Depends on specific construction of jump field $P(\omega, x)$

Exact evaluation possible	Approximation via Fourier inversion

Approximating the stochastic Burgers' equation

Stochastic Burgers' equation

$$u_t + \operatorname{div}\left(a(\omega, x)\frac{u^2}{2}\right) = 0 \qquad \forall (x,t) \in \mathbb{X}_{\mathbb{T}} := (0,1)^2$$

$u(x,0) = u_0(x) = 0.3\sin(\pi x)$

periodic boundary conditions

Approximation of jump coefficient	✓
Spatial approximation	?
Temporal approximation	?
Meshing strategy	?

Numerical approximation

Dafermos (2000), LeVeque (2002)
Adimurthi et al. (2005), Andreianov and Cancès (2012)
Ghoshal et al. (2020), Towers (2000)

Finite Volume method

$$u_j^{m+1} := u_j^m - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{j+\frac{1}{2}} - \mathcal{G}_{j-\frac{1}{2}}\right) \qquad u_j^0 := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\,\mathrm{d}x$$

Generalized Godunov flux (for convex fluxes)

$$\mathcal{G}_{j+\frac{1}{2}} := \mathcal{G}(u, v, x_j, x_{j+1}) = \max\begin{cases} a(\omega, x_j)f(\max(u,0)) \\ a(\omega, x_{j+1})f(\min(v,0)) \end{cases}$$

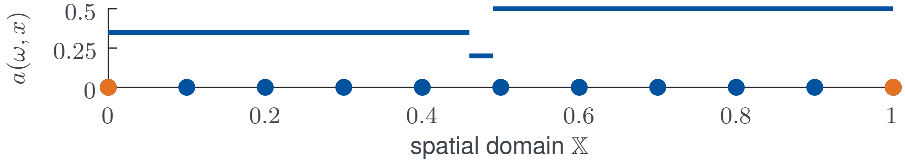
Reduces to classical Godunov flux, if $a(\omega, x_j) = a(\omega, x_{j+1})$

Temporal discretization

Forward Euler scheme
Time step size satisfies CFL condition

Meshing: The naive approach

Equidistant meshing

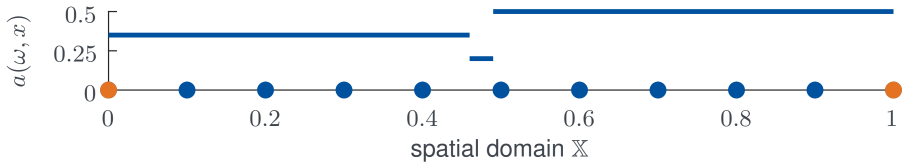


Cell interface at domain boundary

Cell interface resulting from (equidistant) refinement

Meshing: The naive approach

Equidistant meshing



Cell interface at domain boundary

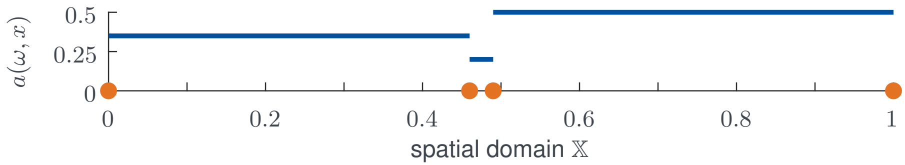
Cell interface resulting from (equidistant) refinement

Does not capture jumps that are close to each other!

$\rightarrow$ **Jump-adaptive meshing**

Jump-adaptive meshing

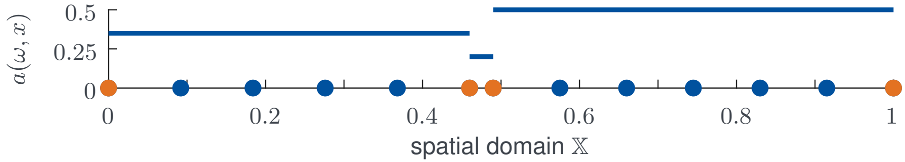
Jump-adaptive meshing Barth, Stein (2020)



Cell interface at jump discontinuity (and domain boundary)

Jump-adaptive meshing

Jump-adaptive meshing Barth, Stein (2020)

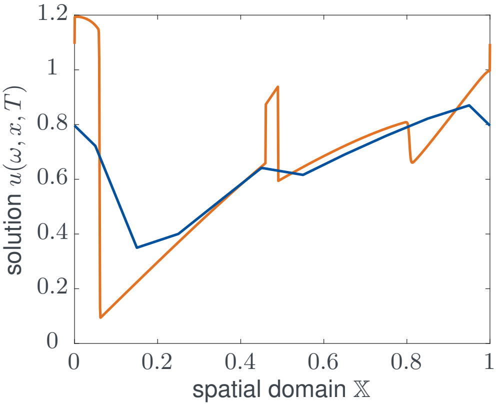


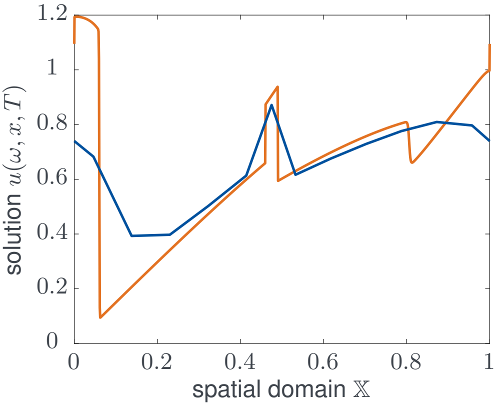
Cell interface at jump discontinuity (and domain boundary)

Cell interface resulting from (piecewise equidistant) refinement

Does capture all jump discontinuities of the coefficient!

Equidistant vs. jump-adaptive meshing

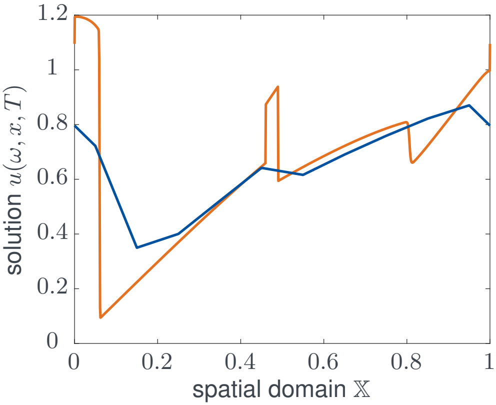


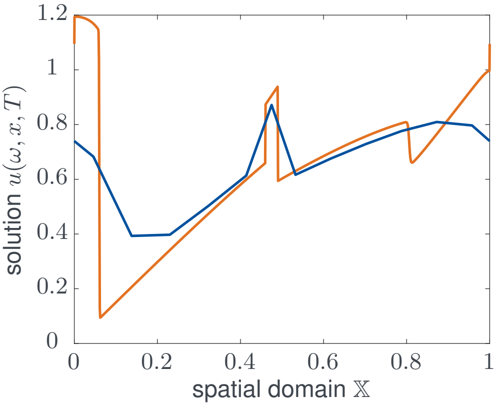


Reference solution with jump-adaptive meshing ($\Delta x \leq 0.001$)

Solution with equidistant (l.) and jump-adaptive (r.) meshing ($\Delta x \leq 0.1$)

Equidistant vs. jump-adaptive meshing



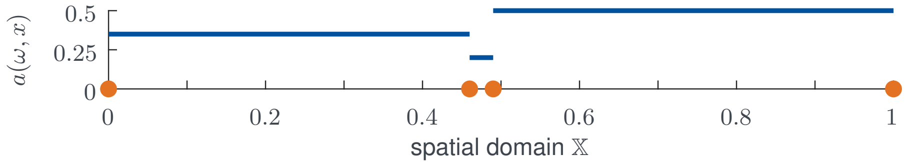


Can *reconstruction error* of standing wave profiles be improved ?

$\rightarrow$ Jump-adaptive wave-cell meshing

Jump-adaptive wave-cell meshing

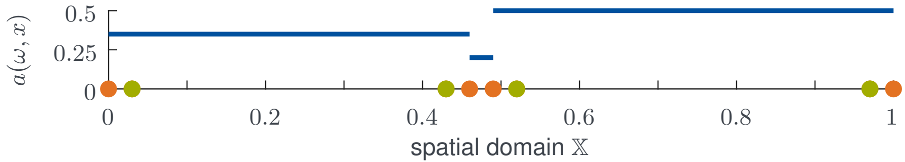
Jump-adaptive wave-cell meshing — **Brencher, Barth (2021)**



Cell interface at jump discontinuity (and domain boundary)

Jump-adaptive wave-cell meshing

Jump-adaptive wave-cell meshing Brencher, Barth (2021)



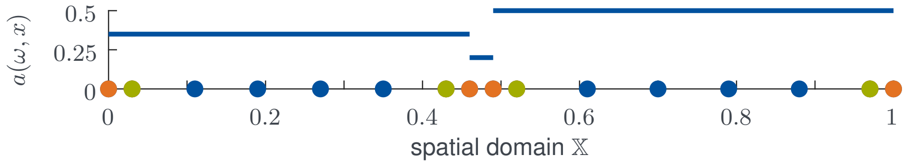
Cell interface at jump discontinuity (and domain boundary)

Cell interfaces of additional wave-cells

Jump-adaptive wave-cell meshing

Brencher and Barth (2021a)

Jump-adaptive wave-cell meshing Brencher, Barth (2021)

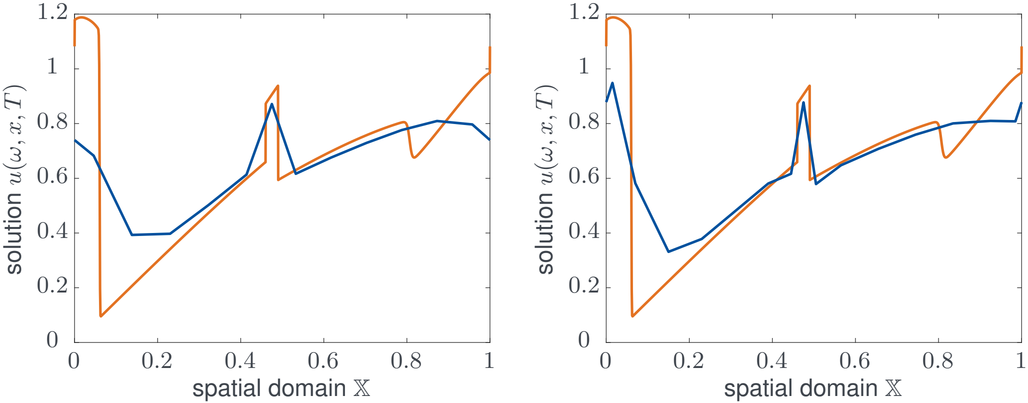


Cell interface at jump discontinuity (and domain boundary)

Cell interfaces of additional wave-cells

Cell interface resulting from (piecewise equidistant) refinement

Jump-adaptive vs. jump-adaptive wave-cell meshing



Reference solution with jump-adaptive wave-cell meshing ($\Delta x \leq 0.001$)

Solution with jump-adaptive (l.) and wave-cell (r.) meshing ($\Delta x \leq 0.1$)

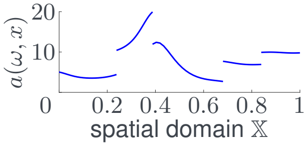
Modeling via stochastic coefficients

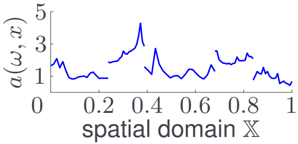
Stochastic Burgers' equation

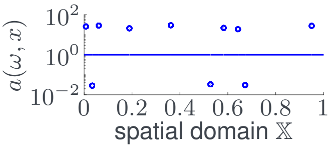
$$u_t + \operatorname{div}\left(a(\omega, x)\frac{u^2}{2}\right) = 0 \qquad \forall (x,t) \in \mathbb{X}_{\mathbb{T}} := (0,1)^2$$

Stochastic jump coefficients

Poisson jump field	Alternating jump field	Jump inclusion field
Smooth Gaussian random field	Rough Gaussian random field	No Gaussian random field







Poisson jump field & smooth GRF

Gaussian random field

Matérn covariance operator

smoothness $\nu = \infty$

variance $\sigma^2 = 0.1$

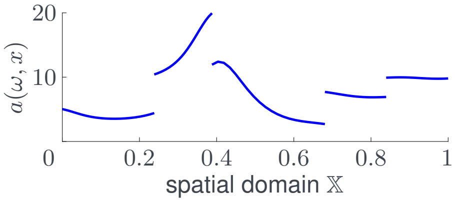
correlation length $\rho = 0.1$

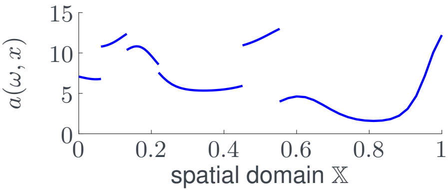
Poisson jump field

number of jumps $\tau \sim \mathrm{Poi}(5) + 1$

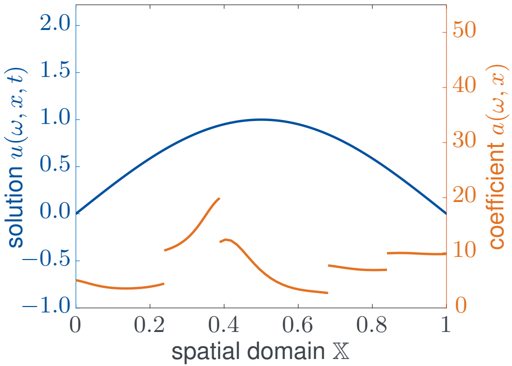
jump locations $\mathfrak{d}_i \sim \mathcal{U}(\mathbb{X})$

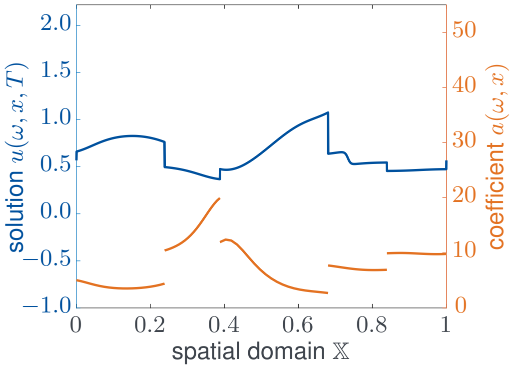
jump heights $P_i \sim \mathrm{Poi}(5) + 1$





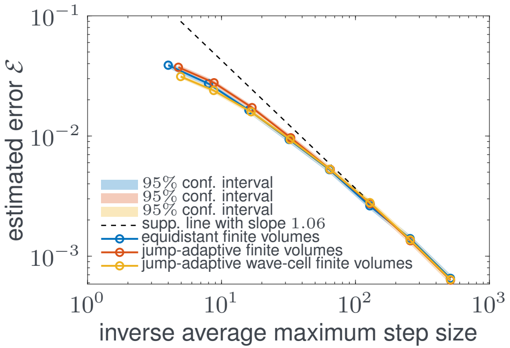
Coefficient sample and corresponding solution



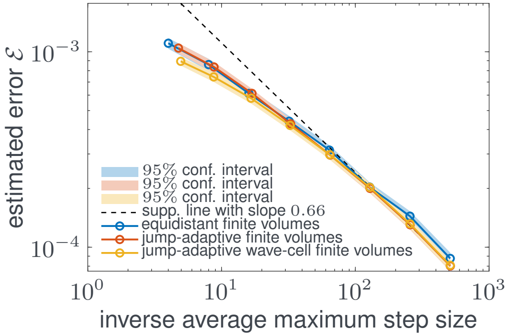


Convergence: Poisson jump field & smooth GRF

Strong error $\mathcal{E}(\omega) := \left\| u_\Delta^{\text{ref}}(\omega, \cdot, T) - u_\Delta(\omega, \cdot, T) \right\|_{L^\star(\mathbb{X})}$



(a) Pathwise L^1 error (50 samples).



(b) Pathwise L^2 error (50 samples).

Alternating jump field & rough GRF

Gaussian random field

Matérn covariance operator

smoothness $\nu = \frac{1}{2}$

variance $\sigma^2 = 0.1$

correlation length $\rho = 0.1$

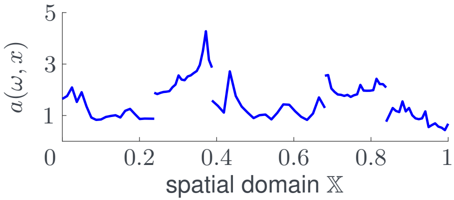
Alternating jump field

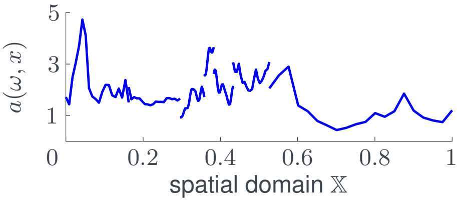
number of jumps $\tau \sim \mathrm{Poi}(5) + 1$

jump locations $\mathfrak{d}_i \sim \mathcal{U}(\mathbb{X})$

jump heights

$$P_i \sim \begin{cases} \mathcal{U}([\frac{1}{4}, \frac{3}{4}]) & \textbf{for } i \textbf{ odd}, \\ \mathcal{U}([\frac{5}{4}, \frac{7}{4}]) & \textbf{for } i \textbf{ even}. \end{cases}$$





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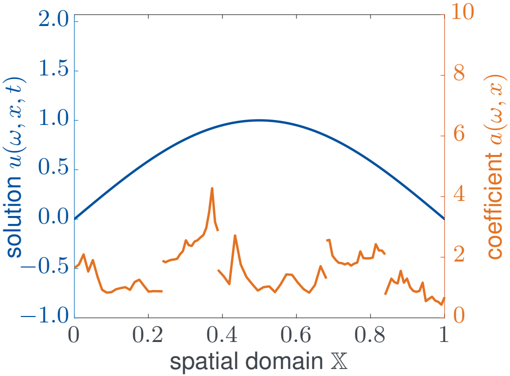

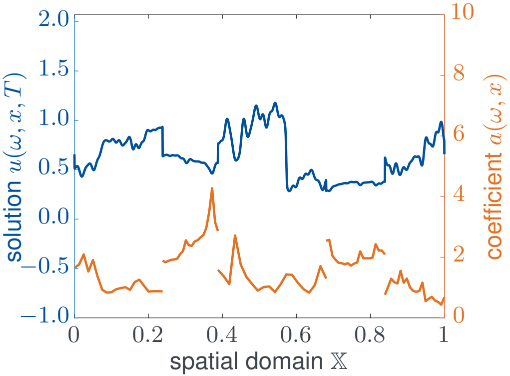

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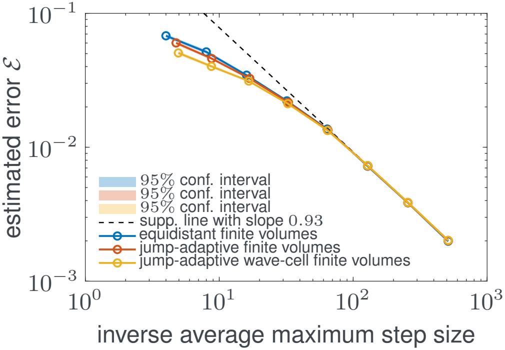

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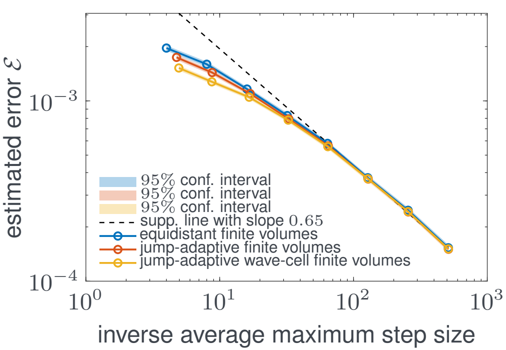


(b) Pathwise L^2 error (50 samples).

Jump inclusion field

Jump inclusion field

set of all inclusions: $\mathfrak{I} \subset \mathcal{D}$

$$a(\omega, x) = \begin{cases} H_k & \text{for } x \in \mathfrak{I}, \\ 1 & \text{otherwise.} \end{cases}$$

inclusion height distribution selected by Bernoulli-distributed random variable $k \sim B(1, \frac{1}{2})$

Inclusion properties

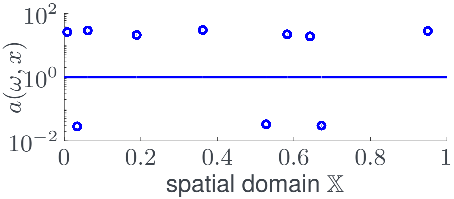
number of inclusions $\tau \sim \mathrm{Poi}(10)$

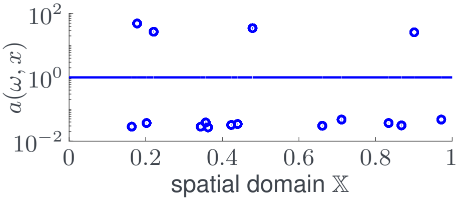
inclusion positions $\chi_i \sim \mathcal{U}(\mathcal{D})$

inclusion sizes $\delta_i \sim \mathcal{U}[10^{-5}, 10^{-3}]$

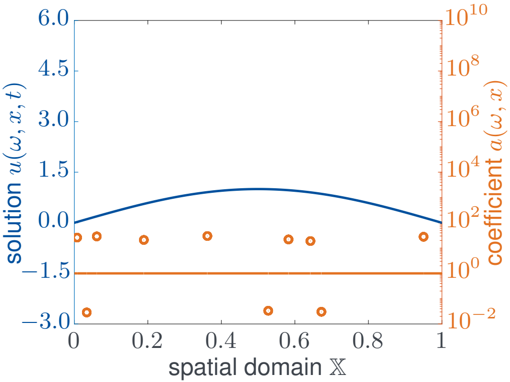
inclusion heights

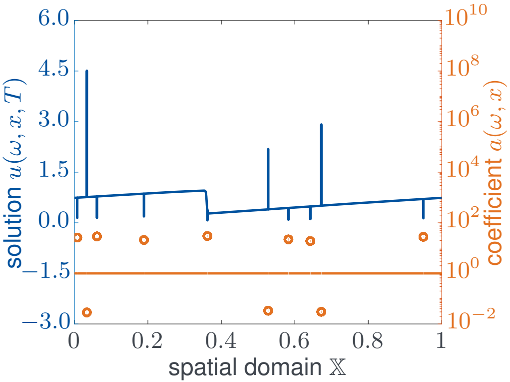
$$H_i \sim \begin{cases} \mathrm{Poi}(30) & \text{for } i = 0 \\ 1/\xi, \xi \sim Poi(30) & \text{for } i = 1 \end{cases}$$





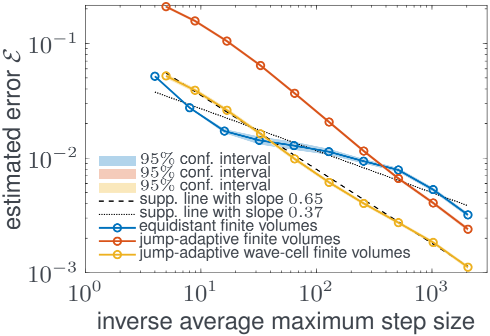
Coefficient sample and corresponding solution



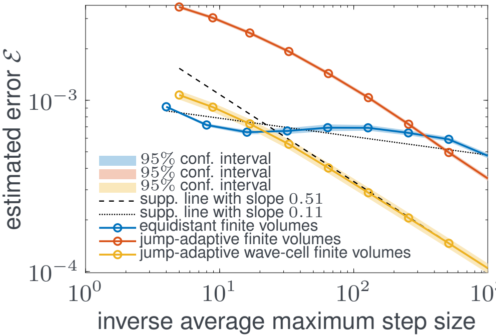


Convergence: Jump inclusion field

Strong error $\mathcal{E}(\omega) := \left\| u_\Delta^{\text{ref}}(\omega,\cdot,T) - u_\Delta(\omega,\cdot,T) \right\|_{L^\star(\mathbb{X})}$



(a) Pathwise L^1 error (100 samples).



(b) Pathwise L^2 error (100 samples).

Thank you!

Conclusion

Flexible modeling approach

Broad class of stochastic coefficients

Numerical scheme

Stochastic Burgers' equation
Discontinuous flux function

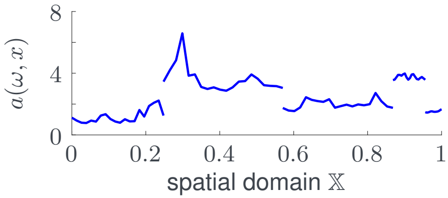
Novel meshing strategy

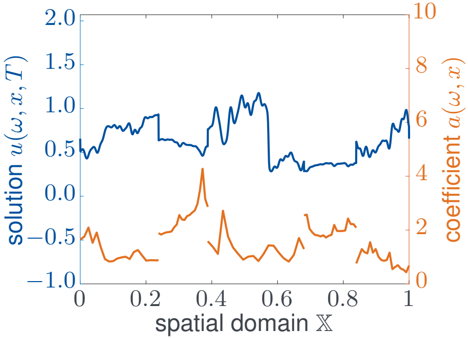
Jump-adaptive wave-cells

Each sample has own discretization

Reduction of samplewise variance

Outperformance of classical meshing strategies





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