University of Stuttgart
Cluster of Excellence in Data-integrated Simulation Science

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Hyperbolic Conservation Laws with Random Discontinuous Fluxes
Analysis and Simulation

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PN5

Motivation: Sub-surface flow & Vehicular traffic

(Scalar) Conservation laws

Insufficient measurements Uncertain permeabilities	**uncertainties**	Weather data Road conditions
Fractured medium Heterogeneous medium	**discontinuities**	Varying speed limits Accidents

$$\begin{aligned} u_t + \operatorname{div}_{\boldsymbol{x}} \mathfrak{f}(\omega, t, \boldsymbol{x}, u) &= 0 \qquad \forall (\omega, t, \boldsymbol{x}) \in \Omega \times \mathbb{R}^d \times (0, T) \\ u(\omega, 0, \boldsymbol{x}) &= u_0(\omega) \qquad \forall (\omega, \boldsymbol{x}) \in \Omega \times \mathbb{R}^d \end{aligned} \tag{1}$$

Well-posedness

- Weak solutions not unique –> additional entropy condition necessary for uniqueness
- Discontinuous flux setting: infinitely many different entropy conditions

Under suitable assumptions:
✓ Existence of pathwise entropy solution
✓ Uniqueness of pathwise entropy solution
✗ **Measurability of entropy solution fails with classical proofs**

Theorem (Measurability of stochastic entropy solutions)

Let $u_0 \in \mathcal{L}^q(\Omega; \mathcal{L}^p(\mathbb{R}^d))$, with $1 \leq q, p \leq \infty$, be a stochastic initial condition to (1). Furthermore, for fixed $\omega \in \Omega$, assume that the solution $u(\omega, \cdot, \cdot)$ takes values in a separable subspace $\mathcal{S} \subset \mathcal{L}^\infty$. Then, the pathwise entropy solution to Problem (1) is strongly measurable in the sense that the mapping $u : \Omega \to \mathcal{S}$ is strongly measurable.

Multiplicative flux & Stochastic jump coefficient

Multiplicative flux function

$$\mathfrak{f}(\omega, t, \boldsymbol{x}, u) = \mathbf{a}(\omega, \boldsymbol{x}) f(u)$$

Stochastic jump coefficient

$$a(\omega, x) := \overline{a}(x) + \phi(W_{\mathcal{D}}(\omega, x)) + P(\omega, x)$$

- $\overline{a} \in C(\mathbb{R}; \mathbb{R}_{\geq 0})$ is a deterministic, uniformly bounded **mean function**.
- $\phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$. In our case: $\phi(w) = \exp(w)$.
- For a **(zero-mean) Gaussian random field** $W \in L^2(\Omega; L^2(\mathbb{R}))$ associated to a non-negative, symmetric trace class (covariance) operator $Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, the random field $W_{\mathcal{D}} \in L^2(\Omega; L^2(\mathbb{R}))$ is defined as
$$W_{\mathcal{D}}(\omega, x) = \begin{cases} W(\omega, x), & x \in \mathcal{D} \\ \min(W(\omega, x), \sup_{x \in \mathcal{D}} W(\omega, x)), & x \in \mathbb{R} \setminus \mathcal{D} \end{cases}$$
- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D})$, $\omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$ is a **random partition of $\mathcal{D}$**, i.e., the $\mathcal{T}_i$ are disjoint open subsets of $\mathcal{D}$ with $\overline{\mathcal{D}} = \bigcup_{i=1}^{\tau} \overline{\mathcal{T}_i}$. The number of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.
For $\mathcal{D}_l$ and $\mathcal{D}_r$ being the left and right boundary of $\mathcal{D}$, respectively, we define $\mathcal{T}_0 := (-\infty, \mathcal{D}_l)$ and $\mathcal{T}_{\tau+1} := (\mathcal{D}_r, +\infty)$.
- $(P_i, i \in \mathbb{N}_0)$ is a sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with arbitrary non-negative distribution(s), which is independent of τ (but not necessarily i.i.d.). Further we have
$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=0}^{\tau+1} \mathbf{1}_{\mathcal{T}_i}(x) P_i(\omega) .$$

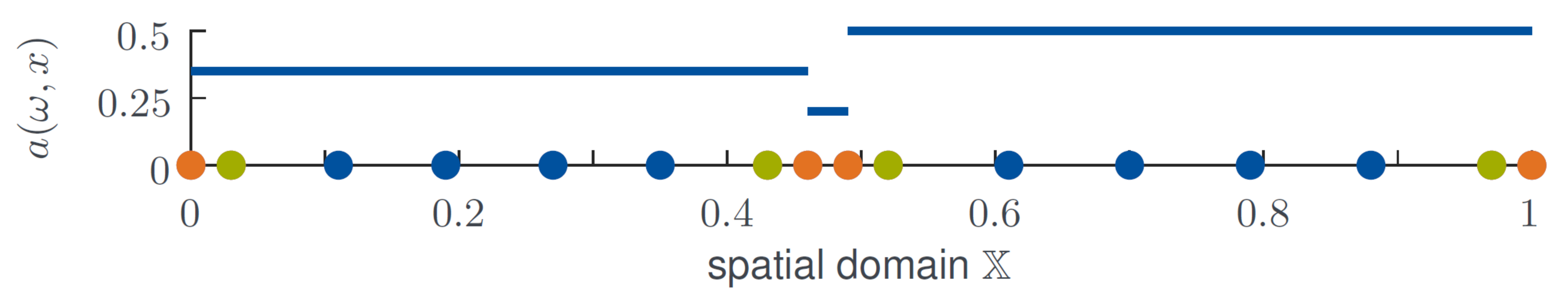
Tools & Methods

- Karhunen-Loève expansion
- Fourier inversion
- Finite Volume method
- Multilevel Monte Carlo method
- Forward Euler method
- Python / Matlab

Numerical approximation & Jump-adapted meshing

- Truncated **Karhunen-Loève** expansion for the Gaussian random field
- Depending on the specific construction of the jump field P: either exact evaluation possible or approximation via **Fourier inversion**.
- **Finite Volume** discretization
- **Forward Euler** scheme satifying the CFL stability condition
- **Godunov flux**: $F_G(u, v) = \max\{f(\max(u, 0)), f(\min(v, 0))\}$

Jump-adapted wave-cell meshing

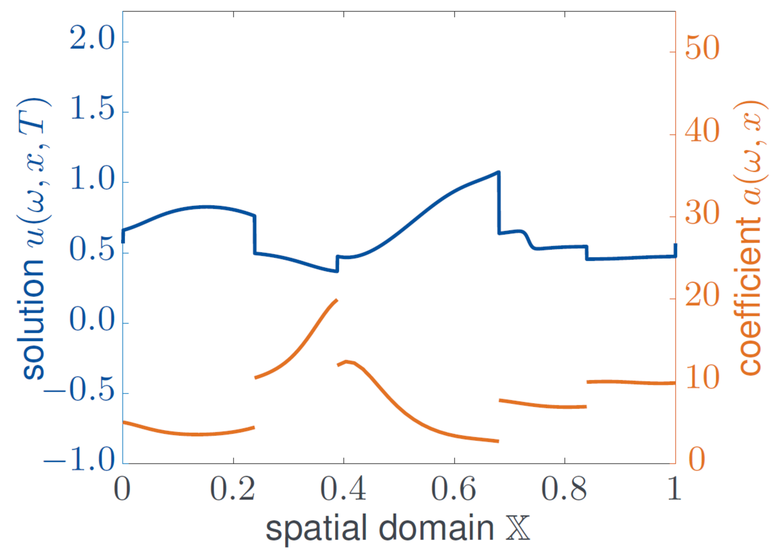


Cell interface at jump discontinuity (and domain boundary)
Cell interfaces of additional wave-cells
Cell interface resulting from (piecewise equidistant) refinement

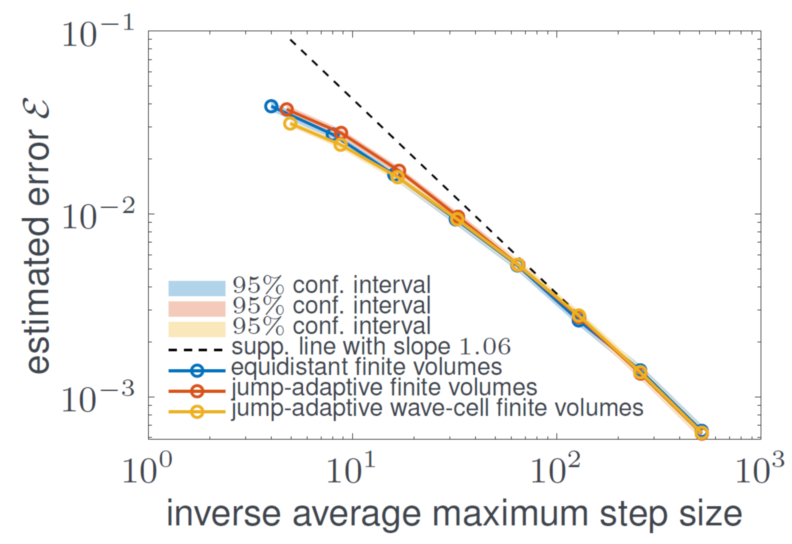
Pathwise solutions & convergence

Stochastic Burger's equation

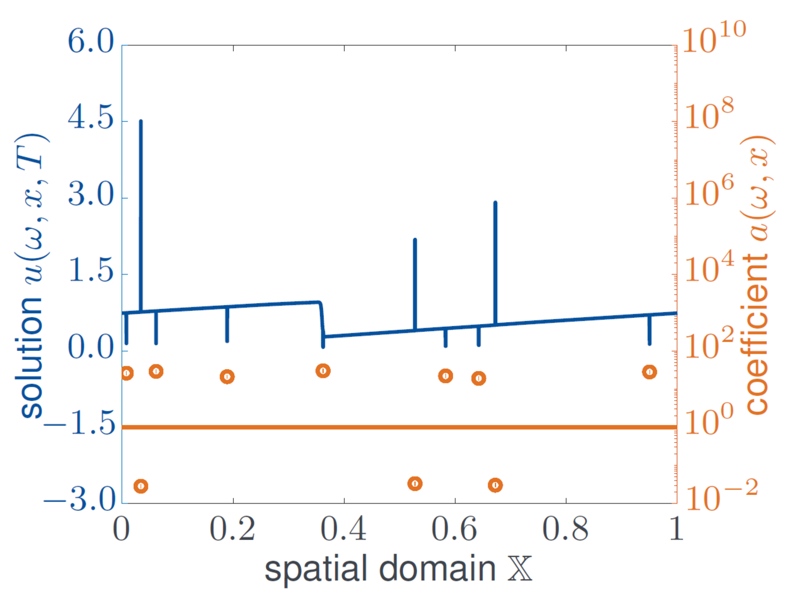
$$u_t + \operatorname{div}\left(a(\omega, x)\frac{u^2}{2}\right) = 0 \qquad \forall (\omega, t, \boldsymbol{x}) \in \Omega \times (0,1) \times (0,1) \tag{2}$$



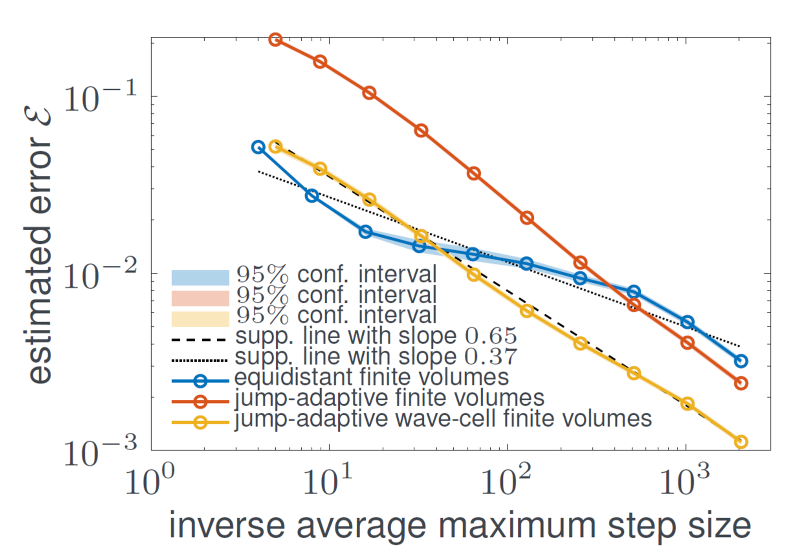
Solution corresponding to a coefficient with smooth Gaussian random field and uniformly distributed jumps.



Strong L^1 error for different meshing strategies in the Finite Volume approximation.



Solution corresponding to a constant coefficient having random inclusions.



Strong L^1 error for different meshing strategies in the Finite Volume approximation

References

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