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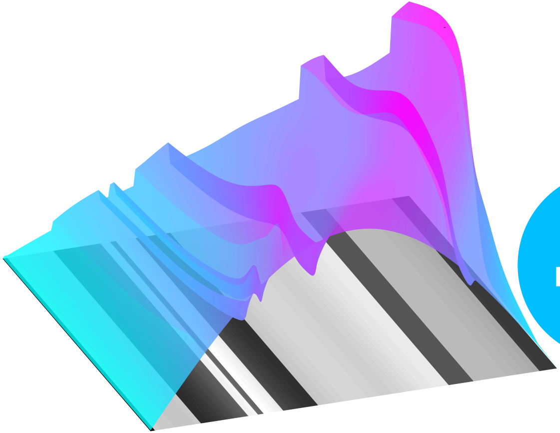

Lukas Brencher

Hyperbolic Conservation Laws with Stochastic Discontinuous Flux

Joint work with Andrea Barth

Talk 15 - FVCA IX

June 18, 2020

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SimTech

Motivation & Problem definition

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Insufficient measurements — Uncertain permeability → Random coefficient

Fractures — Heterogeneities → Random discontinuities

Stochastic conservation law

$(\Omega, \mathcal{A}, \mathbb{P})$ complete probability space.

Unknown $u := u(\omega, x, t)$.

Space-time domain $\mathcal{D}_{\mathbb{T}} := \mathbb{R} \times (0, T)$

$$u_t + (a(\omega, x) f(u))_x = 0$$
$$u(\omega, x, 0) = u_0(\omega, x)$$

$u_0 \in L^p(\Omega; L^\infty(\mathbb{R}))$
Stochastic initial condition

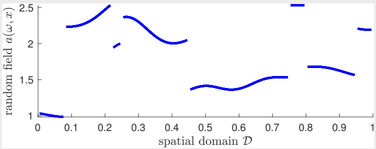
$f(u) \in C^1(\mathbb{R})$
Deterministic flux function

$a : \Omega \times \mathbb{R} \mapsto \mathbb{R}_{>0}$
Stochastic jump coefficient
(possibly time-dependent)

Stochastic jump coefficient

Barth and Stein (2018)

$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x)$$



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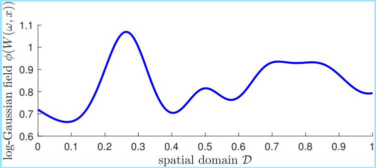
Continuous part $\quad \overline{a}(x) + \phi(W(\omega, x))$



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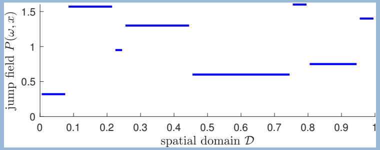
Jump part $\quad P(\omega, x)$



Stochastic jump coefficient

$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x)$$

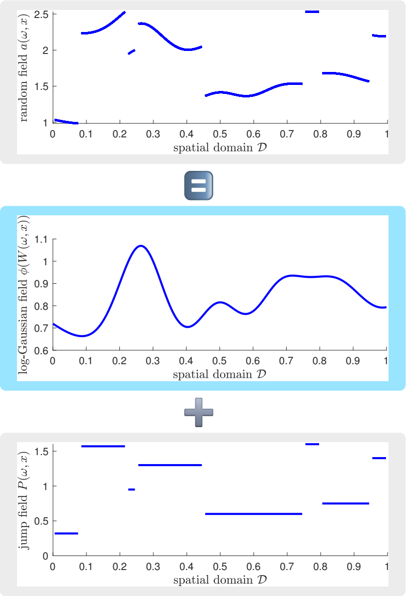
$\overline{a} \in C(\mathbb{R}; \mathbb{R}_{\geq 0})$
Deterministic mean function.

$\phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$
In our case: $\phi(\xi) = \exp(\xi)$.

$W \in L^2(\Omega; L^2(\mathbb{R}))$
(Zero-mean) Gaussian random field

non-negative, symmetric trace class (covariance) operator $Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Barth and Stein (2018)



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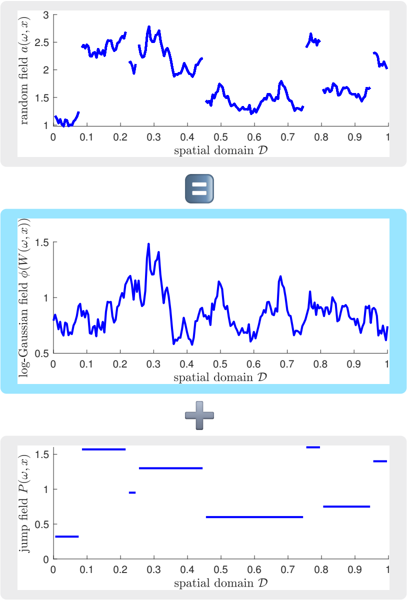


Stochastic jump coefficient

Barth and Stein (2018)

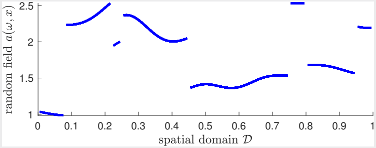
$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x)$$

$\mathcal{T} : \Omega \to \mathcal{B}(\mathbb{R}),\ \omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$
random partition of $\mathbb{R}$.

$\tau : \Omega \to \mathbb{N}$
Random number of elements $\mathcal{T}_i$ in $\mathcal{T}$
λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$
Measure controlling positions of $\mathcal{T}_i$

$$P : \Omega \times \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau} \mathbf{1}_{\mathcal{T}_i}(x) P_i(\omega)$$

$(P_i, i \in \mathbb{N})$
Sequence of non-negative random variables, independent of τ (but not necessarily i.i.d.)



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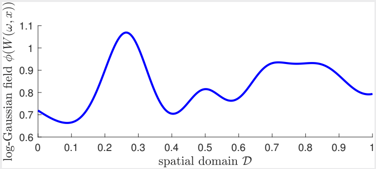

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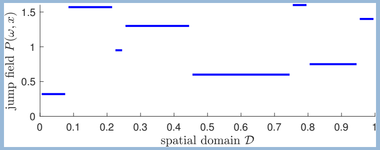

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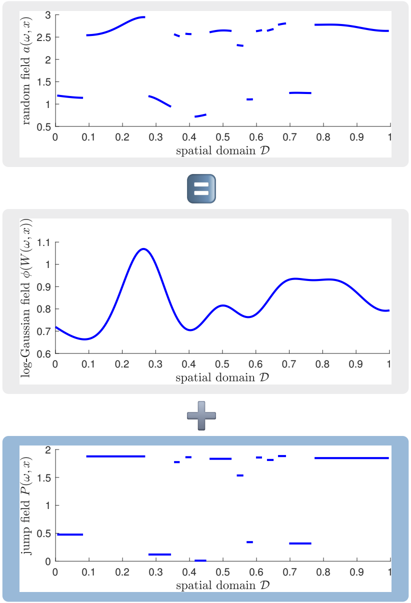


Approximation of the jump coefficient

Gaussian random field $W(\omega, x)$

Karhunen-Loève (KL) expansion

$$W(\omega, x) = \sum_{i=1}^{\infty} \sqrt{\eta_i} e_i(x) Z_i(\omega)$$

$Z_i \sim \mathcal{N}(0, 1)$
Standard-normal random variables

$((\eta_i, e_i), i \in \mathbb{N})$
(ordered) eigenpairs of the covariance operator Q

$\Rightarrow$ **Approximation:** truncate the series after the first $N \in \mathbb{N}$ terms.

Jump field $P(\omega, x)$

Depends on specific construction of jump field $P(\omega, x)$

Exact evaluation possible	Approximation via Fourier inversion

Numerical experiment: Stochastic Burgers' equation

$$u_t + \left(a(\omega, x) \frac{u^2}{2} \right)_x = 0 \qquad \forall (x,t) \in \mathcal{D}_{\mathbb{T}} := (0,1)^2$$

$$u(x,0) = u_0(x) = 0.3 \sin(\pi x)$$

$$u(0,t) = 0 \qquad u_x(1,t) = 0$$

Stochastic jump coefficient $a(\omega, x)$

Continuous part

- Matérn covariance operator with smoothness $\nu \in [\frac{1}{2}, \infty)$
- Gaussian field W sampled via truncated KL expansion

Jump part

- number of jumps: $\tau \sim \mathrm{Poi}(5)$
- jump positions: $x_i \sim \mathcal{U}((0,1))$
- jump heights:

$$P_i \sim \begin{cases} \mathcal{U}([\frac{1}{4}, \frac{3}{4}]), & i \text{ odd} \\ \mathcal{U}([\frac{5}{4}, \frac{7}{4}]), & i \text{ even} \end{cases}$$

Sample-adaptive discretization

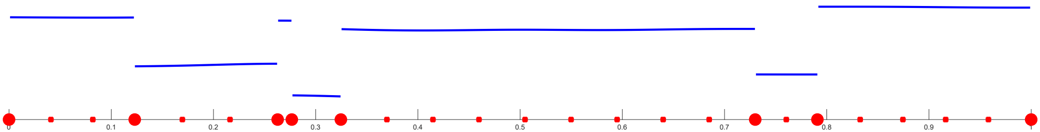
Barth and Stein (2019)

Spatial discretization

Finite Volume discretization with maximum spatial mesh size $\Delta x > 0$.

spatial mesh **adapted** to jump positions $\Rightarrow$ each jump position is cell interface

piecewise equidistant mesh: equidistant between two jumps



Numerical flux & temporal discretization

Adimurthi et al. (2005)
Andreianov and Cancès (2012)
Ghoshal et al. (2020)

Numerical flux

Generalized Godunov flux

$$g(u, v, x_j, x_{j+1}) = \max\{a(\omega, x_j) f(\max(u, 0)), a(\omega, x_{j+1}) f(\min(v, 0))\}$$

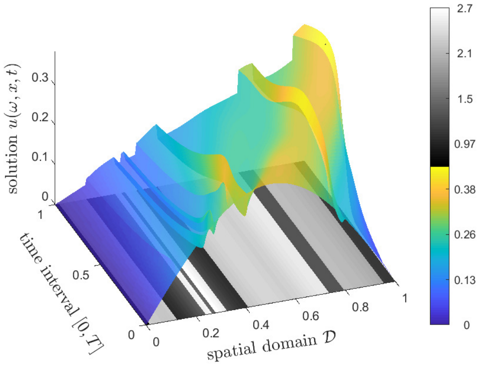
Reduces to classical Godunov flux for $a(\omega, x_j) = a(\omega, x_{j+1})$

Temporal discretization

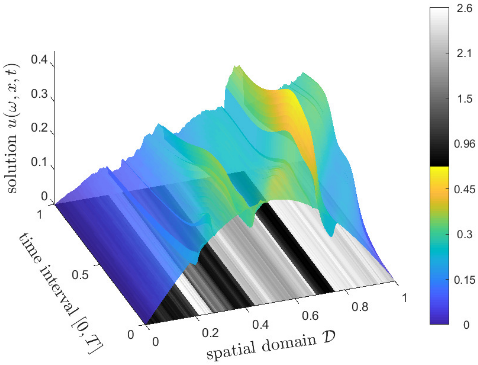
Equidistant time discretization with time step size $\Delta t > 0$

Backward Euler scheme

Realizations of the solution



Squared exponential random field
(smoothness $\nu = \infty$)



Exponential random field
(smoothness $\nu = 1/2$)

Multilevel Monte Carlo

Giles (2015)

Our goal

Approximate stochastic moments (expectation, variance, etc.) of solution

Singlelevel Monte Carlo estimator

$$E_{M_l}(u_l) = \frac{1}{M_l}\sum_{i=1}^{M_l} u_l^{(i)}$$

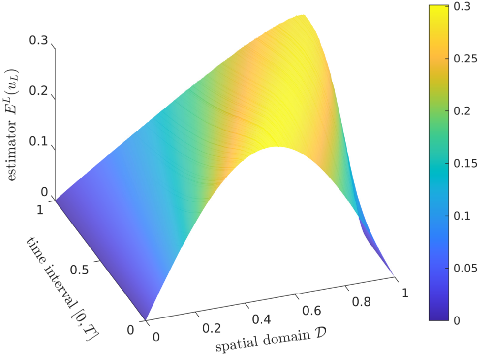
$l \in \mathbb{N}$
Discretization level

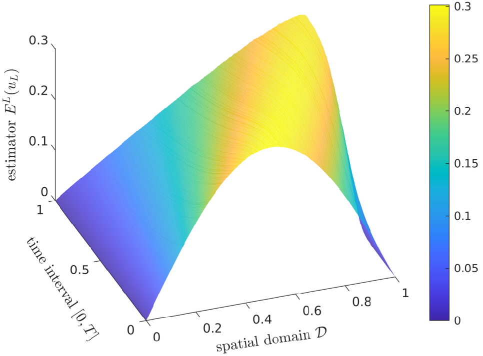
$M_l \in \mathbb{N}$, with $M_{l-1} > M_l$
Number of samples

Multilevel Monte Carlo estimator

$$E^L(u_L) = E_{M_0}(u_0) + \sum_{l=1}^{L} E_{M_l}(u_l - u_{l-1})$$

Multilevel Monte Carlo estimator





Squared exponential random field (smoothness $\nu = \infty$)

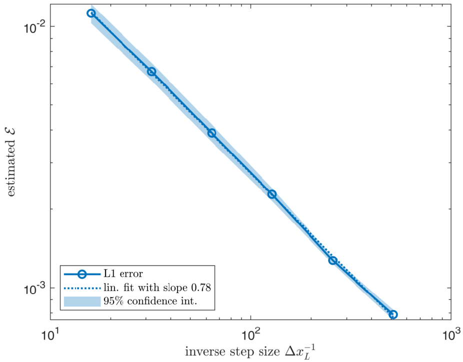
Exponential random field (smoothness $\nu = 1/2$)

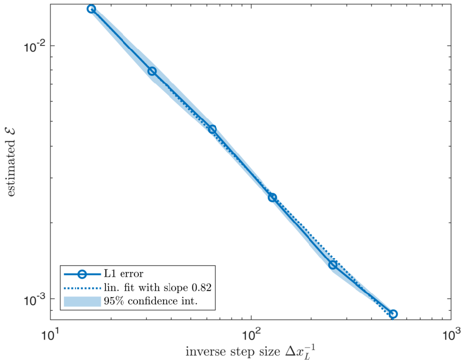
Multilevel Monte Carlo estimation

$$\mathcal{E} := \mathbb{E}\left(\|\mathbb{E}(u) - E^L(u_L)\|_{L^1(\mathcal{D})}\right)$$

All errors aligned in L^1 norm: $\mathcal{E} \simeq \Delta x \simeq \Delta t$

Reference solution via finer discretization





Squared exponential random field

Exponential random field

Thank you!

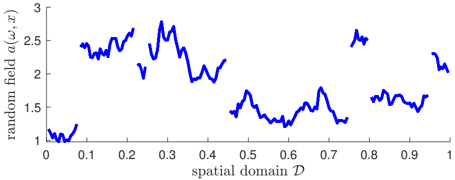
Conclusion

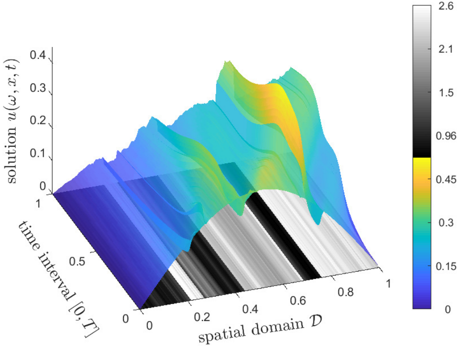
Stochastic conservation law with **discontinuous coefficient**

Sample-adaptive discretization introduced for pathwise convergence

Each sample has its own discretization

Multilevel Monte Carlo estimation of stochastic moments





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