Hyperbolic conservation laws with stochastic jump coefficient
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Motivation - subsurface flows

Description of time-dependent subsurface flows might suffer from:

- Insufficient measurement
- Uncertain permeability

⇒ Random coefficient

Medium might contain:

- Fractures
- Heterogeneities

⇒ Random discontinuities are incorporated
Problem description

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a complete probability space. Consider the scalar hyperbolic conservation law with unknown \(u := u(\omega, x, t)\):

\[
\begin{align*}
    u_t + (a(\omega, x)f(u))_x &= 0 & \forall (x, t) \in \mathcal{D}_T := \mathbb{R} \times (0, T) \\
    u(x, 0) &= u_0(x) & \forall x \in \mathbb{R}
\end{align*}
\]

with

- \(a : \Omega \times \mathcal{D} \mapsto \mathbb{R}\) being a (possibly time-dependent) stochastic jump coefficient,
- \(u_0 : \mathcal{D} \mapsto \mathbb{R}\) being a (possibly stochastic) initial condition.
Jump coefficient

We consider a stochastic jump coefficient of the following structure:

\[ a(\omega, x) := \bar{a}(x) + \phi(W(\omega, x)) + P(\omega, x), \]

where

- \( \bar{a} \in C(D; \mathbb{R}_{\geq 0}) \) is a deterministic mean function.
- \( \phi \in C^1(\mathbb{R}; \mathbb{R}_{>0}) \). In our case: \( \phi(w) = \exp(w) \).
- \( H := L^2(D) \) and \( W \in L^2(\Omega; H) \) is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class (covariance) operator \( Q : H \to H \).
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where

- \( \mathcal{T} : \Omega \rightarrow \mathcal{B}(\mathcal{D}), \omega \mapsto \{\mathcal{T}_1, \ldots, \mathcal{T}_\tau\} \) is a random partition of \( \mathcal{D} \), i.e., the \( \mathcal{T}_i \) are disjoint open subsets of \( \mathcal{D} \) with \( \overline{D} = \bigcup_{i=1}^{\tau} \mathcal{T}_i \). The number of elements in \( \mathcal{T} \) is a random variable \( \tau : \Omega \rightarrow \mathbb{N} \) on \((\Omega, \mathcal{A}, \mathbb{P})\).

- \((P_i, i \in \mathbb{N})\) is a sequence of random variables on \((\Omega, \mathcal{A}, \mathbb{P})\) with arbitrary non-negative distribution(s), which is independent of \( \tau \) (but not necessarily i.i.d.). Further we have

\[ P : \Omega \times \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau} 1_{\mathcal{T}_i}(x) P_i(\omega) \]
Samples of different jump coefficients

Squared exponential Gaussian field with random jumps.

Exponential Gaussian field with random jumps.
Approximation of the jump coefficient

In most cases, the coefficient $a(\omega, x)$ needs to be approximated:

- The Gaussian field $W$ admits the Karhunen-Loève (KL) expansion

$$W(\omega, x) = \sum_{i=1}^{\infty} \sqrt{\eta_i} e_i(x) Z_i(\omega), \quad Z_i \sim \mathcal{N}(0, 1)$$

where $((\eta_i, e_i), i \in \mathbb{N})$ are the (ordered) eigenpairs of the covariance operator $Q$ with $\eta_i \geq 0$ and $e_i \in H$.

- **Approximation:** truncate the series after the first $N \in \mathbb{N}$ terms.

- Note: The number of terms $N$ needed for approximation depends on the decay of the eigenvalues. Therefore, it can vary significantly for different covariance operators.
### Numerical example in 1D - Burgers’ equation

#### Stochastic Burgers’ equation

\[
    u_t + \left( a(\omega, x) \frac{u^2}{2} \right)_x = 0 \quad \forall (x, t) \in \mathcal{D}_T := (0, 1)^2
\]

\[
    u(x, 0) = u_0(x) = 0.3 \sin(\pi x) \quad \forall x \in (0, 1)
\]

#### Stochastic jump coefficient \( a(\omega, x) \)

- Partition \( \mathcal{T} \) is generated by \( \tau \sim \text{Poi}(5) \) jumps with positions \( x_i \sim \mathcal{U}((0, 1)) \)

- Jump heights \( P_i \sim \mathcal{U} \left( \left[ \frac{3}{4} + \frac{1}{2}(-1)^i, \frac{5}{4} + \frac{1}{2}(-1)^i \right] \right) = \begin{cases} \mathcal{U}\left([\frac{1}{4}, \frac{3}{4}]\right), & i \text{ odd} \\ \mathcal{U}\left([\frac{5}{4}, \frac{7}{4}]\right), & i \text{ even} \end{cases} \)

- Squared exponential Gaussian field \( W \) sampled via truncated KL expansion
Numerical discretization

Spatial discretization:

- **Finite Volume discretization** with maximum spatial mesh size $\Delta x > 0$.
- spatial mesh is **adapted** to the jump positions, i.e., at each jump position is a cell interface.
- equidistant discretization between two jumps $\rightarrow$ **piecewise equidistant mesh**.
Numerical discretization

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Temporal discretization:
- Equidistant time discretization $\{t_i\}_{i=0}^{M} \subset \mathbb{T}$ with time step size $\Delta t > 0$
- **Backward Euler** scheme
Numerical flux

Grid points not corresponding to jump positions

- Numerical flux: \( a(\omega, x) g(u, v) \), where \( g(u, v) \) is the \textbf{Godunov flux}:

\[
g(u, v) = \begin{cases} 
\min_{w \in [v, u]} f(w), & v \leq u \\
\max_{w \in [u, v]} f(w), & v \geq u 
\end{cases} = \max\{f(\max(u, 0)), f(\min(v, 0))\}
\]

- Simplification possible due to \( f(u) = \frac{u^2}{2} \).

Grid points at jump positions

- Numerical flux: \textbf{Godunov interface flux}:

\[
g(u, v, a^-(x), a^+(x)) = \max\{a^-(x)f(\max(u, 0)), a^+(x)f(\min(v, 0))\}
\]

- Here, \( a^-(x) \) and \( a^+(x) \) denote the left and right limit of \( a(x) \), respectively.
Numerical results

All errors are aligned in the $L^1$ norm: $\varepsilon \simeq \Delta x \simeq \Delta t$.

Solution of the Burgers’ equation with underlying random jump coefficient.

Convergence of the finite volume discretization.
Conclusion & Consequences

Conclusion

● We introduced a random jump coefficient to a scalar conservation law.
● For pathwise convergence an adaptive discretization was introduced.

⇒ Each sample has its own discretization

Consequences

● Problem: Estimation of solution via (multilevel) Monte Carlo needs (nested) equivalent grids.
● Solution: Introduce finer reference grid, on which every sample is projected. This enables estimation via standard Monte Carlo for samples.


