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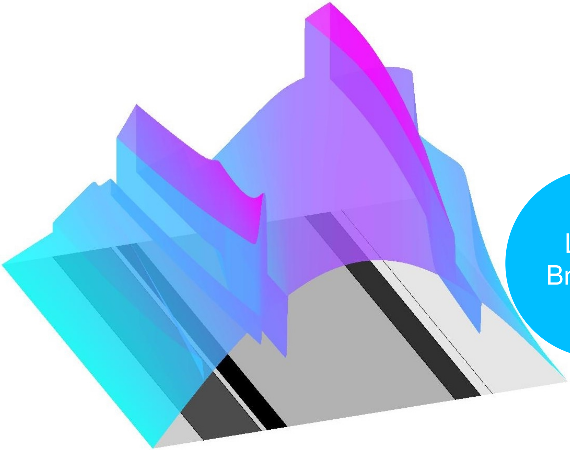

Hyperbolic conservation laws with stochastic jump coefficient

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Joint work with Andrea Barth

NumHyp 2019

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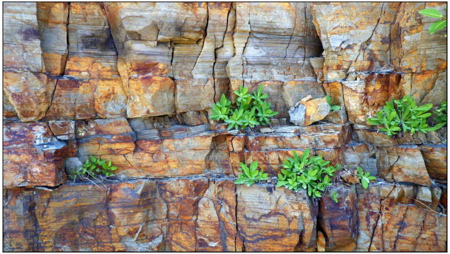
SimTech

Motivation - subsurface flows

Description of time-dependent subsurface flows might suffer from:

- Insufficient measurement
- Uncertain permeability

$\Rightarrow$ **Random coefficient**



Medium might contain:

- Fractures
- Heterogenities

$\Rightarrow$ **Random discontinuities are incorporated**

Problem description

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space.
Consider the scalar hyperbolic conservation law with unknown $u := u(\omega, x, t)$:

$$
\begin{aligned}
u_t + (a(\omega, x) f(u))_x &= 0 \qquad && \forall (x,t) \in \mathcal{D}_{\mathbb{T}} := \mathbb{R} \times (0, T) \\
u(x, 0) &= u_0(x) && \forall x \in \mathbb{R}
\end{aligned}
$$

with

- $a : \Omega \times \mathcal{D} \mapsto \mathbb{R}$ being a (possibly time-dependent) stochastic jump coefficient,
- $u_0 : \mathcal{D} \mapsto \mathbb{R}$ being a (possibly stochastic) initial condition.

Jump coefficient

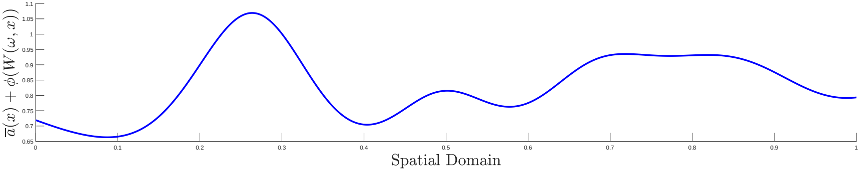
Barth and Stein (2018)

We consider a stochastic jump coefficient of the following structure:

$$a(\omega, x) := \overline{a}(x) + \phi(W(\omega, x)) + P(\omega, x) \ ,$$

where

- $\overline{a} \in C(\mathcal{D}; \mathbb{R}_{\geq 0})$ is a deterministic mean function.
- $\phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$. In our case: $\phi(w) = \exp(w)$.
- $H := L^2(\mathcal{D})$ and $W \in L^2(\Omega; H)$ is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class (covariance) operator $Q : H \to H$.



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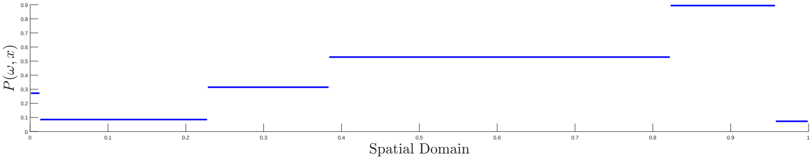
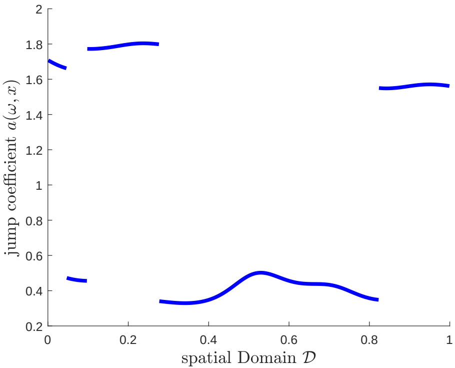
where

- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D}),\ \omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$ is a **random partition of $\mathcal{D}$**, i.e., the $\mathcal{T}_i$ are disjoint open subsets of $\mathcal{D}$ with $\overline{\mathcal{D}} = \bigcup_{i=1}^{\tau} \overline{\mathcal{T}_i}$. The number of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.
- $(P_i, i \in \mathbb{N})$ is a sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with arbitrary non-negative distribution(s), which is independent of τ (but not necessarily i.i.d.). Further we have

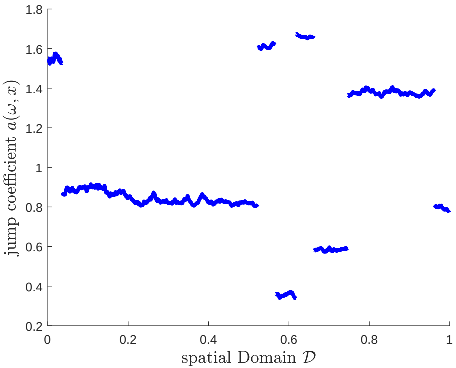
$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau} \mathbf{1}_{\mathcal{T}_i}(x) P_i(\omega)$$



Samples of different jump coefficients



Squared exponential Gaussian field
with random jumps.



Exponential Gaussian field
with random jumps.

Approximation of the jump coefficient

In most cases, the coefficient $a(\omega, x)$ needs to be approximated:

- The Gaussian field W admits the **Karhunen-Loève (KL) expansion**

$$W(\omega, x) = \sum_{i=1}^{\infty} \sqrt{\eta_i} e_i(x) Z_i(\omega) \ , \qquad Z_i \sim \mathcal{N}(0,1)$$

  where $((\eta_i, e_i), i \in \mathbb{N})$ are the (ordered) eigenpairs of the covariance operator Q with $\eta_i \geq 0$ and $e_i \in H$.

- **Approximation:** truncate the series after the first $N \in \mathbb{N}$ terms.

- Note: The number of terms N needed for approximation depends on the decay of the eigenvalues. Therefore, it can vary significantly for different covariance operators.

Numerical example in 1D - Burgers' equation

Stochastic Burgers' equation

$$u_t + \left(a(\omega, x)\frac{u^2}{2}\right)_x = 0 \qquad \forall (x,t) \in \mathcal{D}_{\mathbb{T}} := (0,1)^2$$

$$u(x,0) = u_0(x) = 0.3\sin(\pi x) \qquad \forall x \in (0,1)$$

Stochastic jump coefficient $a(\omega, x)$

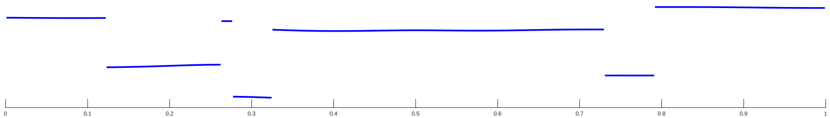
- Partition $\mathcal{T}$ is generated by $\tau \sim Poi(5)$ jumps with positions $x_i \sim \mathcal{U}((0,1))$
- Jump heights $P_i \sim \mathcal{U}\left([\frac{3}{4} + \frac{1}{2}(-1)^i, \frac{5}{4} + \frac{1}{2}(-1)^i]\right) = \begin{cases} \mathcal{U}([\frac{1}{4}, \frac{3}{4}]), & i \text{ odd} \\ \mathcal{U}([\frac{5}{4}, \frac{7}{4}]), & i \text{ even} \end{cases}$
- Squared exponential Gaussian field W sampled via truncated KL expansion

Numerical discretization

Barth and Stein (2019b)

Spatial discretization:

- **Finite Volume discretization** with maximum spatial mesh size $\Delta x > 0$.
- spatial mesh is **adapted** to the jump positions, i.e., at each jump position is a cell interface
- equidistant discretization between two jumps $\rightarrow$ **piecewise equidistant mesh**



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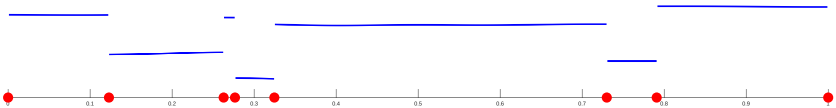

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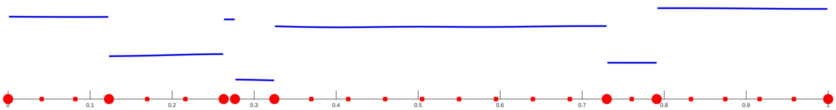

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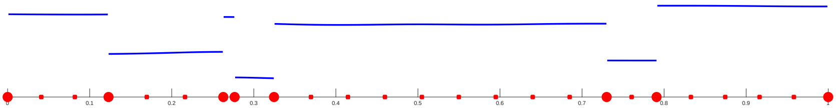


Temporal discretization:

- Equidistant time discretization $\{t_i\}_{i=0}^{M} \subset \mathbb{T}$ with time step size $\Delta t > 0$
- **Backward Euler** scheme

Numerical flux

Adimurthi, Jaffré and Gowda (2004), LeVeque (1992), Towers (2000)
Andreianov and Cancès (2012), Bürger, García, Karlsen and Towers (2008)

Grid points not corresponding to jump positions

- Numerical flux: $a(\omega, x)g(u,v)$, where $g(u,v)$ is the **Godunov flux**:

$$g(u,v) = \left\{ \begin{matrix} \min\limits_{w\in[v,u]} f(w), & v \leq u \\ \max\limits_{w\in[u,v]} f(w), & v \geq u \end{matrix} \right\} = \max\{f(\max(u,0)), f(\min(v,0))\}$$

- simplification possible due to $f(u) = \frac{u^2}{2}$

Grid points at jump positions

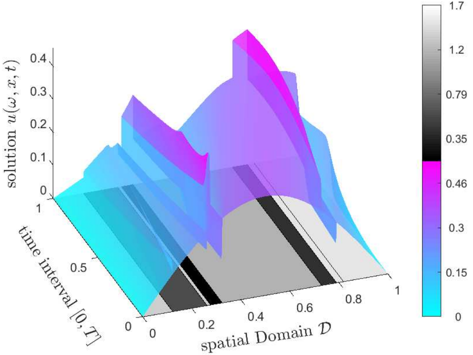
- Numerical flux: **Godunov interface flux**:

$$g(u,v,a^-(x),a^+(x)) = \max\{a^-(x)f(\max(u,0)), a^+(x)f(\min(v,0))\}$$

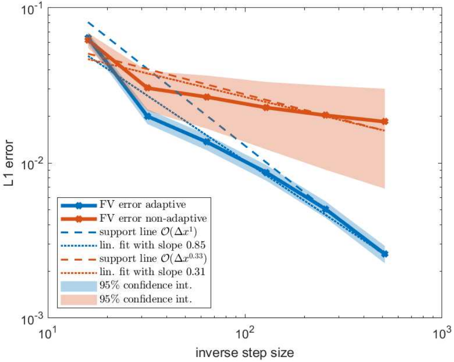
- Here, $a^-(x)$ and $a^+(x)$ denote the left and right limit of $a(x)$, respectively.

Numerical results

All errors are aligned in the L^1 norm: $\varepsilon \simeq \Delta x \simeq \Delta t$.



Solution of the Burgers' equation with underlying random jump coefficient.



Convergence of the finite volume discretization.

Conclusion & Consequences

Conclusion

- We introduced a random jump coefficient to a scalar conservation law.
- For pathwise convergence an adaptive discretization was introduced.

$\Rightarrow$ **Each sample has its own discretization**

Consequences

- Problem: Estimation of solution via (multilevel) Monte Carlo needs (nested) equivalent grids.
- Solution: Introduce finer *reference grid*, on which every sample is projected. This enables estimation via standard Monte Carlo for samples.

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