Hyperbolic conservation laws with stochastic jump coefficient

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Motivation & problem description

When modelling time-dependent subsurface flows, the description might suffer from insufficient measurements or uncertain permeability. To additionally address fractures or heterogeneities of the medium, we incorporate a **random coefficient containing discontinuities** into the description.
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For $\mathcal{D} \subset \mathbb{R}$ a compact domain and $\mathbb{T} := [0, T] \subset \mathbb{R}$, $T \in \mathbb{R}_{>0}$ a time interval consider:

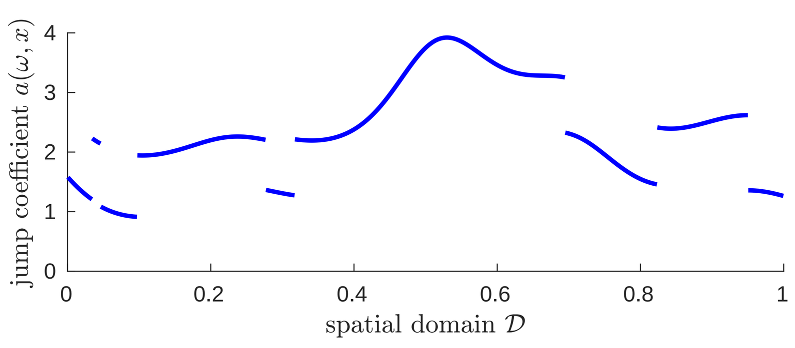
Stochastic Burgers' equation

$$
\begin{aligned}
u_t + (a(\omega, x)\frac{u^2}{2})_x &= 0 && \forall (x,t) \in \mathbb{R} \times \mathbb{T} \\
u(x, 0) = u_0(x) &\in L^\infty(\mathbb{R}) && \forall x \in \mathbb{R}
\end{aligned}
$$

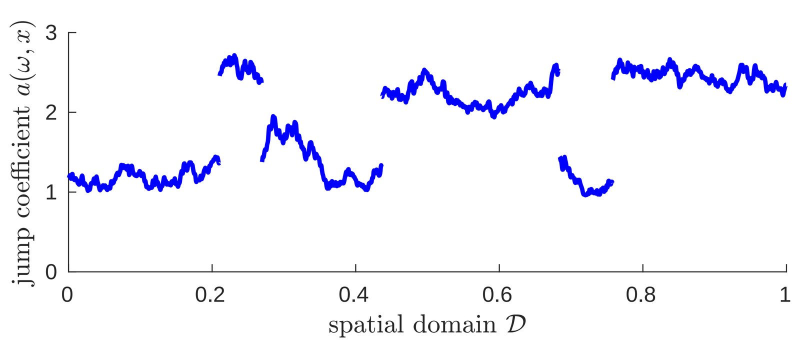
where we consider a stochastic jump coefficient of the form [2, 3]:

$$a(\omega, x) := \overline{a}(x) + \phi(W_\mathcal{D}(\omega, x)) + P(\omega, x)$$

- $\overline{a} \in C(\mathbb{R}; \mathbb{R}_{\geq 0})$ is a deterministic, uniformly bounded **mean function**.
- $\phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$. In our case: $\phi(w) = \exp(w)$.
- For a **(zero-mean) Gaussian random field** $W \in L^2(\Omega; L^2(\mathbb{R}))$ associated to a non-negative, symmetric trace class (covariance) operator $Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, the random field $W_\mathcal{D} \in L^2(\Omega; L^2(\mathbb{R}))$ is defined as
$$W_\mathcal{D}(\omega, x) = \begin{cases} W(\omega, x), & x \in \mathcal{D} \\ \min(W(\omega, x), \sup_{x \in \mathcal{D}} W(\omega, x)), & x \in \mathbb{R} \setminus \mathcal{D} \end{cases}$$
- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D})$, $\omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$ is a **random partition of $\mathcal{D}$**, i.e., the $\mathcal{T}_i$ are disjoint open subsets of $\mathcal{D}$ with $\overline{\mathcal{D}} = \bigcup_{i=1}^{\tau} \overline{\mathcal{T}_i}$. The number of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.
For $\mathcal{D}_l$ and $\mathcal{D}_r$ being the left and right boundary of $\mathcal{D}$, respectively, we define $\mathcal{T}_0 := (-\infty, \mathcal{D}_l)$ and $\mathcal{T}_{\tau+1} := (\mathcal{D}_r, +\infty)$.
- $(P_i, i \in \mathbb{N}_0)$ is a sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with arbitrary non-negative distribution(s), which is independent of τ (but not necessarily i.i.d.). Further we have
$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=0}^{\tau+1} \mathbf{1}_{\mathcal{T}_i}(x) P_i(\omega) \ .$$



Squared exponential Gaussian field with random jumps.



Exponential Gaussian field with random jumps.

The Gaussian field W admits the Karhunen-Loève expansion. Depending on the specific construction of the jump field P, we can either evaluate it exactly or we have to approximate it using Fourier inversion.

Pathwise well-posedness

The flux function $\mathfrak{f}(\omega, x, u) = a(\omega, x)\frac{u^2}{2}$ satisfies the following properties:

- (A-1) $\mathfrak{f}(\omega, \cdot, \cdot)$ is $\mathbb{P}$-a.s. continuous at all points of $\mathbb{R} \setminus \mathcal{N} \times \mathbb{R}$, where $\mathcal{N}$ is a closed zero measure set.
- (A-2) There exist two functions $g_-, g_+ \in L^2(\Omega, C^0(\mathbb{R}))$ such that for all $x \in \mathbb{R}$ it holds $\mathbb{P}$-a.s. that $g_-(\omega, u) \leq |\mathfrak{f}(\omega, x, u)| \leq g_+(\omega, u)$, where g_- is a non-negative (non-strictly) decreasing then increasing function with $|g_-(\omega, \pm\infty)| = +\infty$.
- (A-3) There exists a function $u_M(x) : \mathbb{R} \to \mathbb{R}$ such that for $x \in \mathbb{R} \setminus \mathcal{N}$, $\mathfrak{f}(\omega, x, \cdot)$ is $\mathbb{P}$-a.s. a locally Lipschitz one-to-one function from $(-\infty, u_M(x)]$ and $[u_M(x), +\infty)$ to $[0, +\infty)$ that satisfies $\mathfrak{f}(\omega, x, u_M(x)) = 0$.

For $\mathbb{P}$-a.e. $\omega \in \Omega$, existence and uniqueness of a pathwise weak entropy solution $u(\omega, \cdot, \cdot)$ is proved similarly as for the deterministic hyperbolic problem [1, 4].

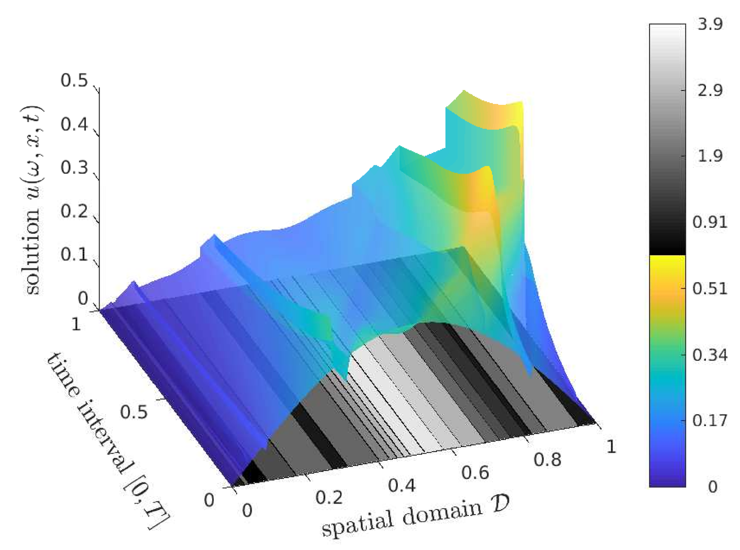
Numerical results

We consider the stochastic Burgers' equation on $\mathcal{D} = [0, 1]$ with $T = 1$ and $u_0 = 0.3\sin(\pi x)$. The continuous part of the random field is given by $\overline{a} \equiv 0$, where the Gaussian field W is characterized by the *Matérn covariance operator* with smoothness parameter either $\nu = \infty$ or $\nu = \frac{1}{2}$, variance $\sigma^2 = 0.1$ and correlation length $\rho = 0.1$.
The partition $\mathcal{T}$ is generated by $\tau \sim \text{Poi}(5) + 2$ resulting in at least one discontinuity of the random field. The jump positions are given by $\varkappa_i \sim \mathcal{U}((0, 1))$ with jump heights $P_i \sim \mathcal{U}([\frac{3}{4} + (-1)^i\frac{1}{2}, \frac{5}{4} + (-1)^i\frac{1}{2}]) = \begin{cases} \mathcal{U}([\frac{1}{4}, \frac{3}{4}]) & i \text{ odd} \\ \mathcal{U}([\frac{5}{4}, \frac{7}{4}]) & i \text{ even} \end{cases}$.

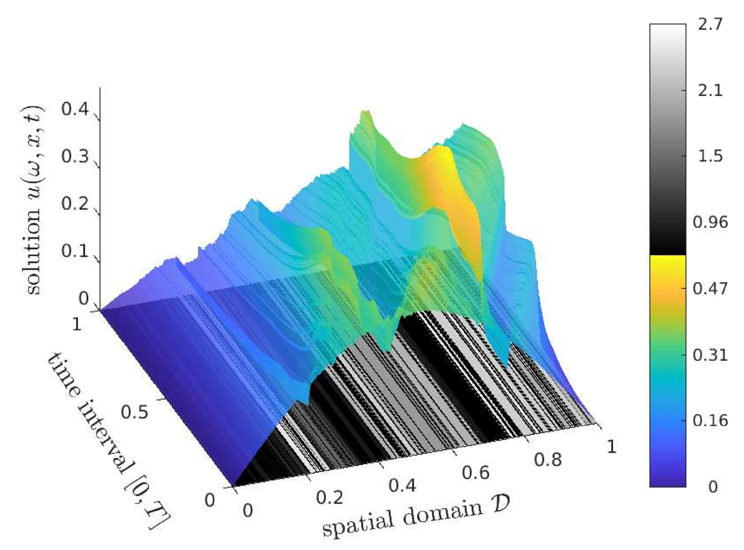
Spatial & temporal discretization

- **Finite Volume** discretization with equidistant spatial mesh size $\Delta x > 0$.
- Equidistant time discretization $\{t_i\}_{i=0}^{M} \subset \mathbb{T}$ with time step size $\Delta t > 0$
- **Backward Euler** scheme
- **Numerical Flux**: Set $f(u) = \frac{u^2}{2}$. On grid points where $a(\omega, x)$ is continuous, we use the classical Godunov flux:

$$g(u, v) = \begin{Bmatrix} \min\limits_{w \in [v,u]} f(w), & v \leq u \\ \max\limits_{w \in [u,v]} f(w), & v \geq u \end{Bmatrix} = \max\{f(\max(u, 0)), f(\min(v, 0))\}$$



Solution of the Burgers' equation with underlying squared-exponential Gaussian field with stochastic jumps.



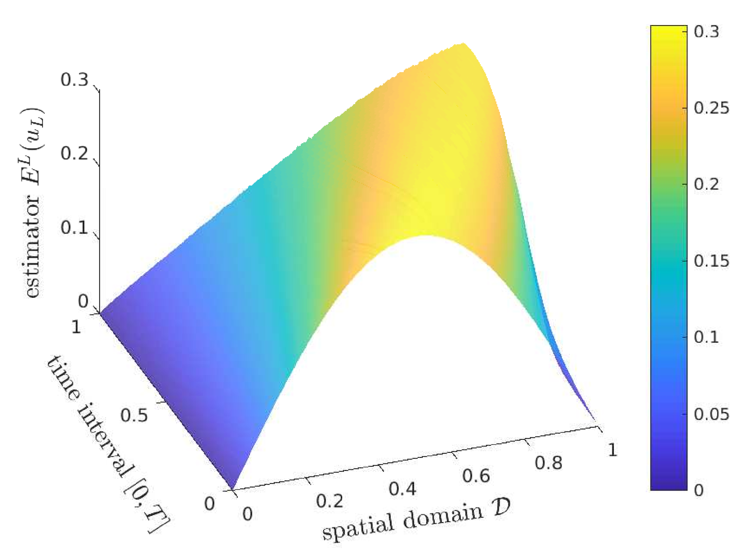
Solution of the Burgers' equation with underlying exponential Gaussian field with stochastic jumps.

Multilevel Monte Carlo

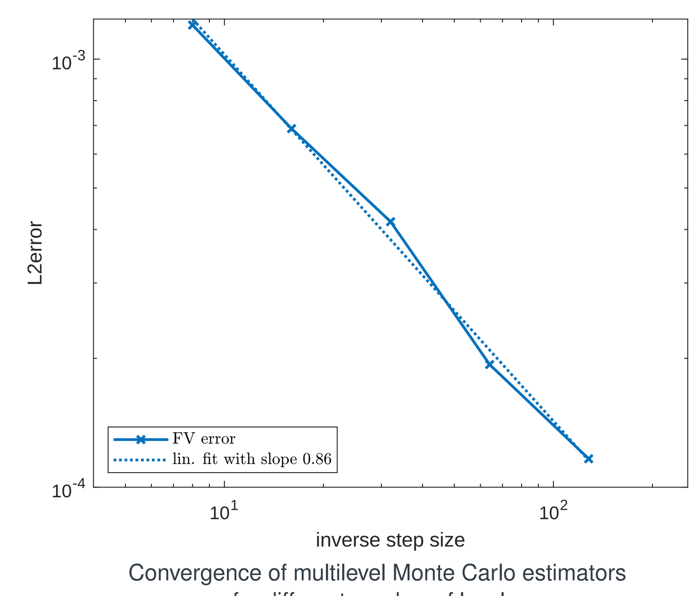
Let $(u_l, l \in \mathbb{N})$ be a sequence of discretizations converging to the exact solution. For given discretization level $l \in \mathbb{N}$ and number of samples $M_l \in \mathbb{N}$ let E_{M_l} denote a Monte Carlo estimator with M_l samples. We aim to approximate the stochastic moments (expectation, variance, etc.) of the solution via the Multilevel Monte Carlo estimator of $\mathbb{E}(u_L)$

$$E^L(u_L) = E_{M_0}(u_0) + \sum_{l=1}^{L} E_{M_l}(u_l - u_{l-1}) \ .$$

Here, $M_0 > \dots > M_L$ are the number of samples computed on each level.



Multilevel Monte Carlo estimator with $L = 5$ levels.



Convergence of multilevel Monte Carlo estimators for different number of levels.

References

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