Hyperbolic conservation laws with stochastic jump coefficient

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Motivation

random coefficient containing discontinuities

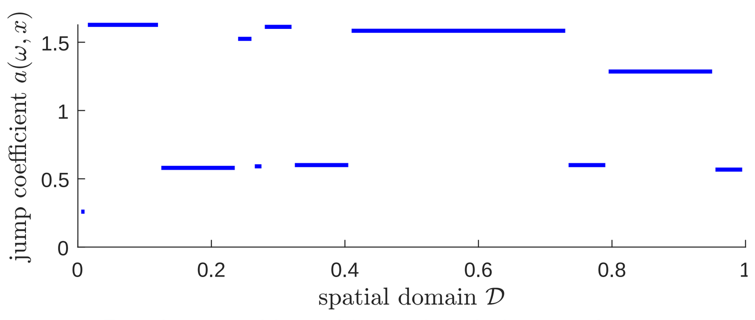
Stochastic conservation law

$$u_t + (\mathfrak{f}(\omega, x, u))_x = 0 \qquad \forall(\omega, x, t) \in \Omega \times \mathbb{R} \times [0, T]$$

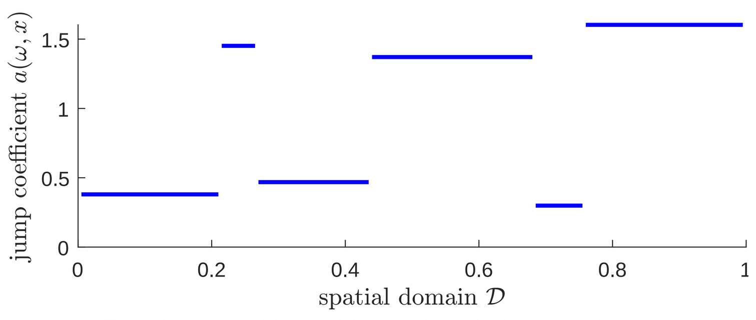
$$u(\cdot, 0) = u_0 \in L^\infty(\mathbb{R})$$

We assume that the flux function $\mathfrak{f}$ has the form $\mathfrak{f}(\omega, x, u) = a(\omega, x) f(u)$.

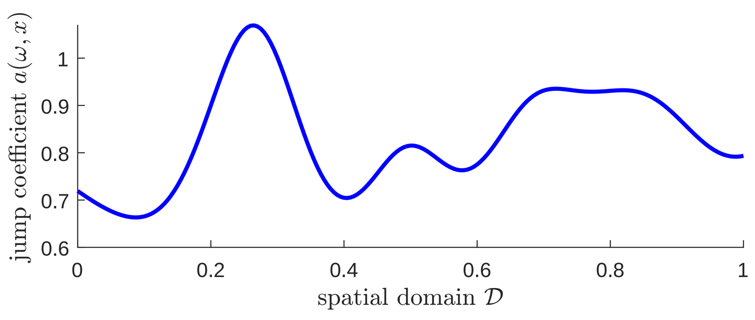
Stochastic jump coefficient



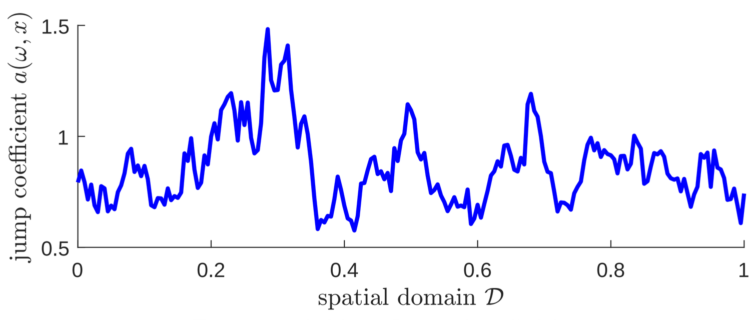
Random partition with stochastic jump values.



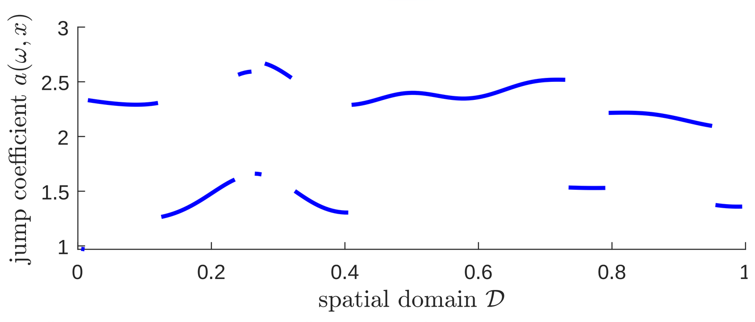
Random partition with stochastic jump values.



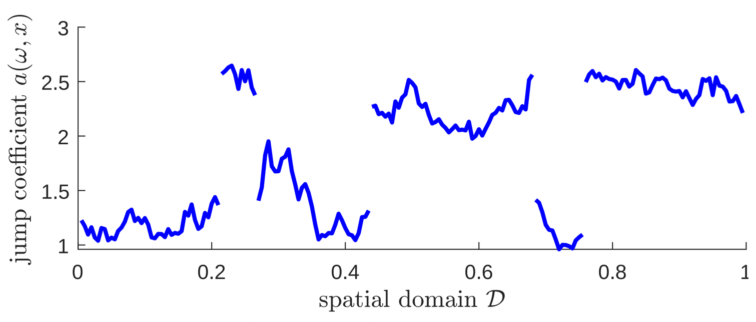
Squared exponential log-Gaussian field.



Exponential log-Gaussian field.



Squared exponential Gaussian field with random jumps.



Exponential Gaussian field with random jumps.

$$a(\omega, x) := \overline{a}(x) + \phi(W_{\mathcal{D}}(\omega, x)) + P(\omega, x)$$

- $\overline{a} \in C(\mathbb{R}; \mathbb{R}_{\geq 0})$ is a deterministic, uniformly bounded mean function.
- $\phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$. In our case: $\phi(w) = \exp(w)$.
- For a (zero-mean) Gaussian random field $W \in L^2(\Omega; L^2(\mathbb{R}))$ associated to a non-negative, symmetric trace class (covariance) operator $Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, the random field $W_{\mathcal{D}} \in L^2(\Omega; L^2(\mathbb{R}))$ is defined as

$$W_{\mathcal{D}}(\omega, x) = \begin{cases} W(\omega, x), & x \in \mathcal{D} \\ \min(W(\omega, x), \sup_{x \in \mathcal{D}} W(\omega, x)), & x \in \mathbb{R} \setminus \mathcal{D} \end{cases}$$

- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D})$, $\omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$ is a random partition of $\mathcal{D}$, i.e., the $\mathcal{T}_i$ are disjoint open subsets of $\mathcal{D}$ with $\overline{\mathcal{D}} = \bigcup_{i=1}^{\tau} \overline{\mathcal{T}_i}$. The number of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{A}, \mathbb{P})$. For $\mathcal{D}_l$ and $\mathcal{D}_r$ being the left and right boundary of $\mathcal{D}$, respectively, we define $\mathcal{T}_0 := (-\infty, \mathcal{D}_l)$ and $\mathcal{T}_{\tau+1} := (\mathcal{D}_r, +\infty)$.
- $(P_i, i \in \mathbb{N}_0)$ is a sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with arbitrary non-negative distribution(s), which is independent of τ (but not necessarily i.i.d.). Further we have

$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=0}^{\tau+1} \mathbf{1}_{\mathcal{T}_i}(x) P_i(\omega) .$$

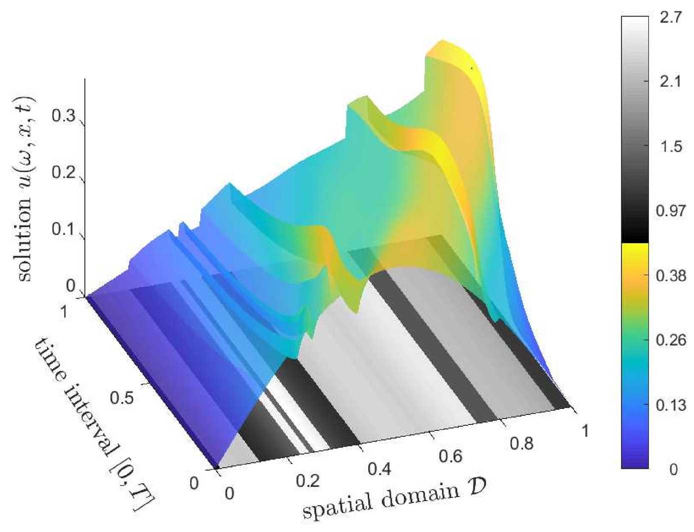
Pathwise well-posedness

- (A-1) $\mathfrak{f}(\omega, \cdot, \cdot)$ is $\mathbb{P}$-a.s. continuous at all points of $\mathbb{R} \setminus \mathcal{N}(\omega) \times \mathbb{R}$, where $\mathcal{N}(\omega)$ is a closed set with measure zero.
- (A-2) There exist two functions $g_-, g_+ \in L^2(\Omega, C^0(\mathbb{R}))$ such that for all $x \in \mathbb{R}$ it holds $\mathbb{P}$-a.s. that $g_-(\omega, u) \leq |\mathfrak{f}(\omega, x, u)| \leq g_+(\omega, u)$, where g_- is a non-negative (non-strictly) decreasing then increasing function with $|g_-(\omega, \pm\infty)| = +\infty$.
- (A-3) There exists a function $u_m : \mathbb{R} \to \mathbb{R}$ such that for $x \in \mathbb{R} \setminus \mathcal{N}(\omega)$, $\mathfrak{f}(\omega, x, \cdot)$ is $\mathbb{P}$-a.s. a locally Lipschitz one-to-one function from $(-\infty, u_m(x)]$ and $[u_m(x), +\infty)$ to $[0, +\infty)$ such that $\mathfrak{f}(\omega, x, u_m(x)) = 0$.
- (A-3') For $x \in \mathbb{R} \setminus \mathcal{N}(\omega)$, the flux function $\mathfrak{f}(\omega, x, \cdot)$ is $\mathbb{P}$-a.s. a locally Lipschitz one-to-one function from $\mathbb{R}$ to $\mathbb{R}$.

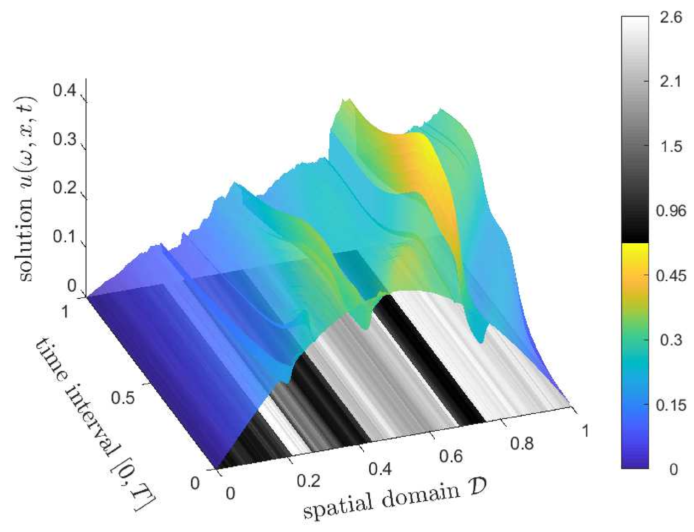
Under the assumptions (A-1) – (A-3), or alternatively (A-1) – (A-3'), there exists $\mathbb{P}$-a.s. a unique pathwise adapted entropy solution to the Cauchy problem of the stochastic conservation law.

Numerical approximation

- Truncated **Karhunen-Loève** expansion for the Gaussian random field
- Depending on the specific construction of the jump field P: either exact evaluation possible or approximation via **Fourier inversion**.
- **Finite Volume** discretization with equidistant spatial mesh size
- **Backward Euler** scheme on an equidistant time discretization
- **Godunov flux**: $F_G(u, v) = \max\{f(\max(u, 0)), f(\min(v, 0))\}$



Solution of the Burgers' equation with underlying squared-exponential Gaussian field with stochastic jumps.



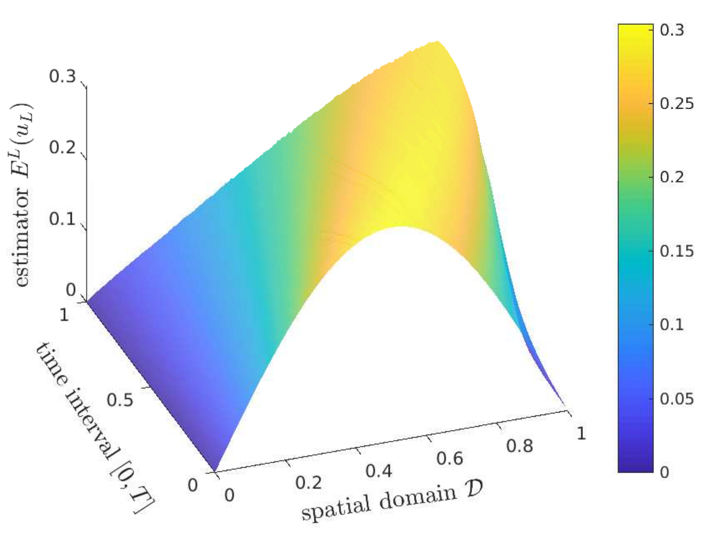
Solution of the Burgers' equation with underlying exponential Gaussian field with stochastic jumps.

Moment approximation

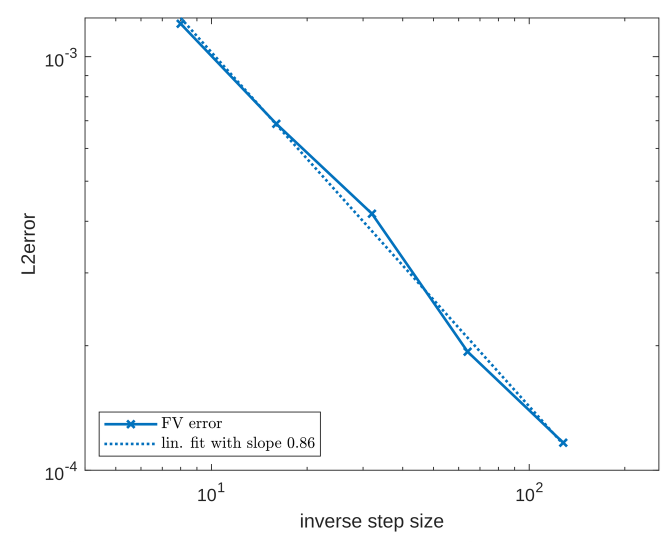
We aim to approximate the stochastic moments (expectation, variance, etc.) of the solution via the Multilevel Monte Carlo estimator of $\mathbb{E}(u_L)$

$$E^L(u_L) = E_{M_0}(u_0) + \sum_{l=1}^{L} E_{M_l}(u_l - u_{l-1}) .$$

Here, $M_0 > \ldots > M_L$ are the number of samples computed on each level.



Multilevel Monte Carlo estimator with $L = 5$ levels.



Convergence of multilevel Monte Carlo estimators for different number of levels.

Tools & Methods

- Karhunen-Loève expansion
- Fourier inversion
- Finite Volume method
- Multilevel Monte Carlo method
- Backward Euler method

MATLAB®

References

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[2] Andrea Barth and Andreas Stein. A study of elliptic partial differential equations with jump diffusion coefficients. *SIAM/ASA Journal on Uncertainty Quantification*, 6(4):1707–1743, 2018.

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[4] Gui-Qiang Chen, Nadine Even, and Christian Klingenberg. Hyperbolic conservation laws with discontinuous fluxes and hydrodynamic limit for particle systems. *Journal of differential equations*, 245(11):3095–3126, 2008.

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