Universität Stuttgart

SimTech
Cluster of Excellence

I. Maier[a] · **B. Haasdonk**[a]

Output error bounds for the Dirichlet-Neumann reduced basis method

Stuttgart, November 2013

[a] Institute of Applied Analysis and Numerical Simulation, University of Stuttgart,
Pfaffenwaldring 57, 70569 Stuttgart/ Germany
{immanuel.maier,haasdonk}@mathematik.uni-stuttgart.de
http://www.mathematik.uni-stuttgart.de/fak8/ians/lehrstuhl/agh/index.html

Abstract The Dirichlet-Neumann reduced basis method is a model order reduction method for homogeneous domain decomposition of elliptic PDE's on a-priori known geometries. It is based on an iterative scheme with full offline-online decomposition and rigorous a-posteriori error estimates. We show that the primal-dual framework for non-compliant output quantities can be transferred to this method. The results are validated by numerical experiments with a thermal block model.

Keywords Reduced basis method · Domain decomposition · Dual solution · Output error · Error estimation

Stuttgart Research Centre for Simulation Technology (SRC SimTech)
SimTech – Cluster of Excellence
Pfaffenwaldring 5a
70569 Stuttgart

publications@simtech.uni-stuttgart.de
www.simtech.uni-stuttgart.de

Output error bounds for the Dirichlet-Neumann reduced basis method

Immanuel Maier[1] and Bernard Haasdonk[2]

[1] University of Stuttgart, Institut of Applied Analysis and Numerical Simulation, Pfaffenwaldring 57, 70569 Stuttgart, Germany, immanuel.maier@mathematik.uni-stuttgart.de
[2] haasdonk@mathematik.uni-stuttgart.de

Abstract. The Dirichlet-Neumann reduced basis method is a model order reduction method for homogeneous domain decomposition of elliptic PDE's on a-priori known geometries. It is based on an iterative scheme with full offline-online decomposition and rigorous a-posteriori error estimates. We show that the primal-dual framework for non-compliant output quantities can be transferred to this method. The results are validated by numerical experiments with a thermal block model.

1 Introduction

Recently, several approaches combining the reduced basis (RB) method — a model reduction method for efficient treatment of parametrized partial differential equations (PDE's) — and domain decomposition — a technique for coupling PDE's on adjacent computational domains — have been developed [2], [4], [3], [1]. A standard RB approach consists in approximating the solution manifold of a parametrized PDE by a low-dimensional linear space spanned by so-called snapshots — highly accurate solutions computed with Finite Elements (FE) for example — and a Galerkin-projection on this space. In a domain decomposition framework it is no longer neccessary to compute detailed solutions on the whole domain. Furthermore, the dimensions of RB approximation spaces on subdomains may be lower than in the monolithic approach.

The Dirichlet-Neumann RB method [1] is based on the well-known Dirichlet-Neumann procedure for FE. It represents an iterative method for linear elliptic problems with an offline/online decomposition, which allows solving the PDE in a very fast online-stage. All high-dimensional FE computations are done in the offline-stage. It also includes effective a-posteriori error estimation for RB approximations, that possibly are discontinuous over the internal boundary.

In this contribution we provide an extension of the method regarding computation and error estimation of output quantities. We make use of the primal-dual framework, which is commonly known to produce good output error bounds for non-compliant problems. We refer to [5] and [6] for an introduction into output error estimation for standard RB methods.

2 Problem definition

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz–boundary $\partial\Omega$ and $x \in \overline{\Omega}$ the space variable. We introduce a Hilbert space $X \subset H_0^1(\Omega)$ with the norm $\|v\|_X := \|v\|_{H^1(\Omega)}$ which can be either finite or infinite dimensional. We now consider a decomposition of Ω into 2 subdomains, i.e. $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The interface Γ is defined as $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. We assume that Ω_1 and Ω_2 have Lipschitz–boundaries and that Γ, $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$ have a nonvanishing $(n-1)$-dimensional measure. Several function spaces are defined according to the domain decomposition.

$$\begin{aligned}
X_k &:= \left\{ v_{|\Omega_k} | v \in X \right\}, \\
X_k^0 &:= \left\{ v \in X_k | \gamma v = 0 \right\}, \\
X_\Gamma &:= \gamma(X_1) = \gamma(X_2),
\end{aligned}$$

where $k = 1, 2$. The operator γ denotes the trace operator on Γ, where we do not notationally discriminate between the spaces X_1 or X_2, as it will always be clear from the context. It holds $X_1 \subset H^1(\Omega_1)$, $X_2 \subset H^1(\Omega_2)$ and $X_\Gamma \subset H_{00}^{1/2}(\Gamma)$. We equip the Hilbert spaces X_k, $k = 1, 2$ with the norms $\|v\|_{X_k} := \|v\|_{H^1(\Omega_k)}$ and X_Γ with $\|g\|_{X_\Gamma} := \|g\|_{L_2(\Gamma)}$.

Now let $\mathcal{P} \subset \mathbb{R}^P$, $P \in \mathbb{N}$ be the domain of the parameter $\mu \in \mathcal{P}$. We introduce the parametric elliptic variational problem for defining the parameter-dependent primal solution $u(\mu) \in X$ and the output $s(\mu) \in \mathbb{R}$:

$$a(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X, \tag{1}$$

$$s(\mu) = l(u(\mu); \mu), \tag{2}$$

with a parametric, symmetric bilinear form $a : X \times X \times \mathcal{P} \to \mathbb{R}$ and parametric linear forms $f, l : X \times \mathcal{P} \to \mathbb{R}$. Furthermore, the so-called dual problem for defining the dual solution $\psi(\mu)$ reads

$$a(v, \psi(\mu); \mu) = a(\psi(\mu), v; \mu) = -l(v; \mu), \quad \forall v \in X. \tag{3}$$

The approximation of $\psi(\mu)$ in the RB scheme helps to get good output approximations, although the dual problem is not neccessary for the computation of $s(\mu)$.

2.1 Assumptions

We assume that, for all $\mu \in \mathcal{P}$, a is continuous on X and coercive on X with coercivity constant

$$\alpha_X(\mu) := \inf_{v \in X \setminus \{0\}} \frac{a(v, v; \mu)}{\|v\|_X^2} > 0.$$

We also assume that f and l are continuous and that a, f and l are parameter separable, i.e. for all of them exist decompositions of the following type:

$$a(v, w; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(v, w), \quad \forall v, w \in X, \mu \in \mathcal{P},$$

with preferably small integer Q_a and μ–independent continuous bilinear forms a^q.

We assume that the solution $u(\mu)$ of (1) is approximated with an iterative domain decomposition procedure. To this end, symmetric bilinear forms $a_k(v, w; \mu) : X_k \times X_k \times \mathcal{P} \to \mathbb{R}$ ("$a|_{\Omega_k}$") and linear forms $f_k(v; \mu) : X_k \times \mathcal{P} \to \mathbb{R}$ ("$f|_{\Omega_k}$") are given on the subdomains. This enables also to define a and f on

$$W := X_1 \oplus X_2,$$

which can be identified with a superset of X. For details we refer the reader to [1]. To complete the notational framework we introduce the continuity constant

$$M_W(\mu) := \sup_{v \in W \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{a(v, w; \mu)}{\|v\|_W \|w\|_W} < \infty.$$

3 Reduced basis scheme

The approximation of the output $s(\mu)$ defined in (2) for a parameter $\mu \in \mathcal{P}$ consists in an offline-stage, which is done once, and an online-stage, which is performed for every output evaluation. In the offline-stage bases for the RB approximation spaces on the subdomains are generated. This is done in a Greedy-algorithm, using a fastly evaluable a-posteriori error estimate to get the "worst-error" parameter. The bases are extended stepwise by a specific routine, yielding partly orthonormalized bases. For more details we refer the reader to [1]. The primal and dual problem are treated equally in this step, yielding separate primal and dual RB spaces. We concentrate now on the explanation of the online-stage, where the approximations to $s(\mu)$ are actually computed.

We introduce RB spaces $X_{N,k} \subset X_k$, $X_{N,k}^0 \subset X_k^0$ and $X_{N,k}^\Gamma \subset X_k$ for $k = 1, 2$ with dimensions $N_k := \dim(X_{N,k}) < \infty$, $N_k^0 := \dim(X_{N,k}^0) < \infty$, $N_k^\Gamma := \dim(X_{N,k}^\Gamma) < \infty$ for $k = 1, 2$ and the following relations:

$$\begin{aligned} X_{N,k} &\cong X_{N,k}^0 \oplus X_{N,k}^\Gamma, \quad k = 1, 2 \\ X_{N,k}^0 &= X_{N,k} \cap H_0^1(\Omega_k), \quad k = 1, 2, \\ \gamma(X_{N,1}^\Gamma) &= \gamma(X_{N,2}^\Gamma), \end{aligned}$$

Consequently, it holds $N_k = N_k^0 + N_k^\Gamma$ for $k = 1, 2$ and $N^\Gamma := N_1^\Gamma = N_2^\Gamma$. Further we define $X_{N,\Gamma} := \gamma(X_{N,1}^\Gamma) = \gamma(X_{N,2}^\Gamma)$. This one-to-one correspondence on the interface allows us to transmit values without evaluating traces in $X_{N,k}$ online. It also enables us to define a lifting operator in the following way:

$$R^X_{N,1} : X_{N,\Gamma} \to X_{N,1} : g \mapsto (\gamma|_{X^\Gamma_{N,1}})^{-1} g.$$

We assume that those spaces were built for the approximation of the primal solution. For the approximation of the dual solution we introduce RB spaces $Y_{N,k} \subset X_k$, $Y^0_{N,k} \subset X^0_k$, $Y^\Gamma_{N,k} \subset X_k$ for $k = 1, 2$ and $Y_{N,\Gamma} \subset X_\Gamma$ with exactly the same properties. The corresponding dimensions are denoted M_k, M^0_k and M^Γ and the lifting operator $R^Y_{N,1} := (\gamma|_{Y^\Gamma_{N,1}})^{-1}$.

Definition 1 (Primal and dual iteration). Given $\mu \in \mathcal{P}$, $g^0_N(\mu) = 0 \in X_{N,\Gamma}$, $\lambda^0_N(\mu) = 0 \in Y_{N,\Gamma}$ and $\theta^n_N(\mu), \eta^n_N(\mu) \in [0, 1]$ for $n \geq 1$. We construct sequences $u^n_{N,1}(\mu) \in X_{N,1}$, $u^n_{N,2}(\mu) \in X_{N,2}$ and $g^n_N(\mu) \in X_{N,\Gamma}$ for $n \geq 1$ satisfying

$$\begin{aligned}
a_1(u^n_{N,1}(\mu), v; \mu) &= f_1(v; \mu), \quad \forall v \in X^0_{N,1}, \\
\gamma u^n_{N,1}(\mu) &= g^{n-1}_N(\mu), \\
a_2(u^n_{N,2}(\mu), v; \mu) &= f_2(v; \mu) + f_1(R^X_{N,1}\gamma v; \mu) \\
&\quad - a_1(u^n_{N,1}(\mu), R^X_{N,1}\gamma v; \mu), \quad \forall v \in X_{N,2}, \\
g^n_N(\mu) &= (1 - \theta^n_N(\mu))\, g^{n-1}_N(\mu) + \theta^n_N(\mu)\gamma u^n_{N,2}(\mu)
\end{aligned}$$

and sequences $\psi^n_{N,1}(\mu) \in Y_{N,1}$, $\psi^n_{N,2}(\mu) \in Y_{N,2}$ and $\lambda^n_N(\mu) \in Y_{N,\Gamma}$ for $n \geq 1$ satisfying

$$\begin{aligned}
a_1(v, \psi^n_{N,1}(\mu); \mu) &= -l_1(v; \mu), \quad \forall v \in Y^0_{N,1}, \\
\gamma \psi^n_{N,1}(\mu) &= \lambda^{n-1}_N(\mu), \\
a_2(v, \psi^n_{N,2}(\mu); \mu) &= -l_2(v; \mu) - l_1(R^Y_{N,1}\gamma v; \mu) \\
&\quad - a_1(R^Y_{N,1}\gamma v, \psi^n_{N,1}(\mu); \mu), \quad \forall v \in Y_{N,2}, \\
\lambda^n_N(\mu) &= (1 - \eta^n_N(\mu))\, \lambda^{n-1}_N(\mu) + \eta^n_N(\mu)\gamma \psi^n_{N,2}(\mu).
\end{aligned}$$

Remark 2. Those infinite sequences are terminated as soon as

$$\begin{aligned}
\|u^n_{N,1}(\mu) - u^{n-1}_{N,1}(\mu)\|^2_{1,\mu} + \|u^n_{N,2}(\mu) - u^{n-1}_{N,2}(\mu)\|^2_{2,\mu} &\leq \epsilon_{\text{tol}}, \\
\|\psi^n_{N,1}(\mu) - \psi^{n-1}_{N,1}(\mu)\|^2_{1,\mu} + \|\psi^n_{N,2}(\mu) - \psi^{n-1}_{N,2}(\mu)\|^2_{2,\mu} &\leq \epsilon_{\text{tol}},
\end{aligned}$$

for some $\epsilon_{\text{tol}} > 0$ where $\|v\|_{k,\mu} := \sqrt{a_k(v, v; \mu)}$ for all $v \in X_k$. The numbers of actually accomplished iterations are denoted by $n_{u,\text{acc}}(\mu)$ and $n_{\psi,\text{acc}}(\mu)$, respectively.

3.1 Smoothed solutions

For $n \geq 1$ we define $u^n_N(\mu) := (u^n_{N,1}(\mu), u^n_{N,2}(\mu)) \in W$. In general $u^n_N(\mu) \notin X$ and so we define $\hat{u}^n_N(\mu) := R_1(\gamma u^n_{N,1}(\mu) - \gamma u^n_{N,2}(\mu))$, where $R_1 : X_\Gamma \to X_1$

is an arbitrary but linear lifting operator, that is $\gamma R_1 g = g$ for all $g \in X_\Gamma$. We get the following representation:

$$u_N^n(\mu) = \hat{u}_N^n(\mu) + \bar{u}_N^n(\mu),$$

with a smoothed solution $\bar{u}_N^n(\mu) := u_N^n(\mu) - \hat{u}_N^n(\mu) \in X$ and a part $\hat{u}_N^n(\mu)$ compensating for the jump on the interface. For $n \to \infty$ the solution $u_N^n(\mu)$ converges to a smooth function [1], so $\hat{u}_N^n(\mu)$ tends to zero.

Analogously we define $\psi_N^n(\mu) := (\psi_{N,1}^n(\mu), \psi_{N,2}^n(\mu)) \in W$, $\hat{\psi}_N^n(\mu) := R_2(\gamma\psi_{N,2}^n(\mu) - \gamma\psi_{N,1}^n(\mu))$, where $R_2 : X_\Gamma \to X_2$ is an arbitrary but linear lifting operator, and $\bar{\psi}_N^n(\mu) = \psi_N^n(\mu) - \hat{\psi}_N^n(\mu) \in X$. As a result, $\bar{\psi}_N^n(\mu)|_{\Omega_1} = \psi_{N,1}^n(\mu)$ in contrast to $\bar{u}_N^n(\mu)|_{\Omega_2} = u_{N,2}^n(\mu)$. This will simplify the offline/online-decomposition of our output approximation.

Definition 3 (Output approximation). Given $\mu \in \mathcal{P}$ and corresponding primal and dual solutions $u_N^{n_u}(\mu)$, $n_u \geq 1$ and $\psi_N^{n_\psi}(\mu)$, $n_\psi \geq 1$ we define the corresponding output approximation

$$s_N^{(n_u,n_\psi)}(\mu) := l(\bar{u}_n^{n_u}(\mu);\mu) + f(\bar{\psi}_N^{n_\psi}(\mu);\mu) - a(\bar{u}_N^{n_u}(\mu), \bar{\psi}_N^{n_\psi}(\mu);\mu). \qquad (4)$$

4 Error estimation

The a-posteriori error estimate of the linear output relies on a-posteriori estimates for the primal and dual solutions. To be more precise, we use estimates for the above defined smoothed solutions $\bar{u}_N^n(\mu)$ and $\bar{\psi}_N^n(\mu)$. To that, we define residuals $r_u^n(\cdot;\mu) \in X'$ and $r_\psi^n(\cdot;\mu) \in X'$ for $n \geq 1$ and $\mu \in \mathcal{P}$ through:

$$r_u^n(v;\mu) := f(v;\mu) - a(u_N^n(\mu), v;\mu), \qquad \forall v \in X,$$
$$r_\psi^n(v;\mu) := -l(v;\mu) - a(v, \psi_N^n(\mu);\mu), \qquad \forall v \in X.$$

Proposition 4. *Given $n \geq 1$ and $\mu \in \mathcal{P}$, the errors $u(\mu) - \bar{u}_N^n(\mu)$ and $\psi(\mu) - \bar{\psi}_N^n(\mu)$ can be estimated in the energy-norm $|||\cdot|||_\mu = \sqrt{a(\cdot,\cdot;\mu)}$ via*

$$|||u(\mu) - \bar{u}_N^n(\mu)|||_\mu \leq \Delta_u^n(\mu),$$
$$|||\psi(\mu) - \bar{\psi}_N^n(\mu)|||_\mu \leq \Delta_\psi^n(\mu),$$

where

$$\Delta_u^n(\mu) := \frac{1}{\sqrt{\alpha_X^{\mathrm{LB}}(\mu)}} \|r_u^n(\cdot;\mu)\|_{X'} + \frac{M_W^{\mathrm{UB}}(\mu)}{\sqrt{\alpha_X^{\mathrm{LB}}(\mu)}} \|\hat{u}_N^n(\mu)\|_{X_1}, \qquad (5)$$

$$\Delta_\psi^n(\mu) := \frac{1}{\sqrt{\alpha_X^{\mathrm{LB}}(\mu)}} \|r_\psi^n(\cdot;\mu)\|_{X'} + \frac{M_W^{\mathrm{UB}}(\mu)}{\sqrt{\alpha_X^{\mathrm{LB}}(\mu)}} \left\|\hat{\psi}_N^n(\mu)\right\|_{X_2}. \qquad (6)$$

Here $\alpha_X^{\mathrm{LB}}(\mu)$ denotes a computable lower bound for the constant $\alpha_X(\mu)$ and $M_W^{\mathrm{UB}}(\mu)$ a computable upper bound for $M_W(\mu)$.

The Proposition 4 for the primal varibale was proven in [1]. The proof for the dual variable follows the same lines.

Corollary 5. *Given* n_u, $n_\psi \geq 1$ *and* $\mu \in \mathcal{P}$, *the error* $|s(\mu) - s_N^{(n_u,n_\psi)}(\mu)|$ *can be estimated via*

$$|s(\mu) - s_N^{(n_u,n_\psi)}(\mu)| \leq \Delta_s^{(n_u,n_\psi)}(\mu),$$

where

$$\begin{aligned}\Delta_s^{(n_u,n_\psi)}(\mu) &= \Delta_u^{n_u}(\mu)\Delta_\psi^{n_\psi}(\mu) \\ &= \frac{1}{\alpha_X^{\mathrm{LB}}(\mu)}\left(\|r_u^{n_u}(\cdot;\mu)\|_{X'} + M_W^{\mathrm{UB}}(\mu)\|\hat{u}_N^{n_u}(\mu)\|_{X_1}\right) \\ &\quad \left(\|r_\psi^{n_\psi}(\cdot;\mu)\|_{X'} + M_W^{\mathrm{UB}}(\mu)\|\hat{\psi}_N^{n_\psi}(\mu)\|_{X_2}\right).\end{aligned}$$

Thanks to the smoothness of the solutions in the output approximation, the proof of Corollary 5 is analogue to the proof for the standard RB method [5], [6].

4.1 Offline/online decomposition

As already mentioned, an efficient offline/online decomposition is essential for our method. The parameter separability (4) is the main ingredient for obtaining such a decomposition. Again we refer to [1] for a detailed explanation of the routine for the primal iteration. Offline/online decomposition of the dual iteration is achieved in the same way. For a decomposition of the output approximation (4) into parameter-dependent coefficients and parameter-independent components we exploit

$$\begin{aligned}s_N^{(n_u,n_\psi)}(\mu) &= l(\bar{u}_n^{n_u}(\mu)) + f(\bar{\psi}_N^{n_\psi}(\mu);\mu) - a(\bar{u}_N^{n_u}(\mu),\bar{\psi}_N^{n_\psi}(\mu);\mu) \\ &= l(u_N^{n_u}(\mu);\mu) - l_1(R_1\gamma u_{N,1}^{n_u}(\mu);\mu) + l_1(R_1\gamma u_{N,2}^{n_u}(\mu);\mu) \\ &\quad + f(\psi_N^{n_\psi}(\mu);\mu) - f_2(R_2\gamma\psi_{N,2}^{n_\psi}(\mu);\mu) + f_2(R_2\gamma\psi_{N,1}^{n_\psi}(\mu);\mu) \\ &\quad - a(u_N^{n_u}(\mu),\psi_N^{n_\psi}(\mu);\mu) \\ &\quad + a_1(R_1\gamma u_{N,1}^{n_u}(\mu),\psi_{N,1}^{n_\psi}(\mu);\mu) - a_1(R_1\gamma u_{N,2}^{n_u}(\mu),\psi_{N,1}^{n_\psi}(\mu);\mu) \\ &\quad + a_2(u_{N,2}^{n_u}(\mu),R_2\gamma\psi_{N,2}^{n_\psi}(\mu);\mu) - a_2(u_{N,2}^{n_u}(\mu),R_2\gamma\psi_{N,1}^{n_\psi}(\mu);\mu).\end{aligned}$$

Details on the offline/online decomposition of the error estimate (5), respectively (6) can also be found in [1].

5 Numerical results

We consider the static heat equation on the unit square in $\mathbb{R}^2$ with a decomposition of the domain into two parts. The heat coefficient $\kappa(\mu)$ is piecewise

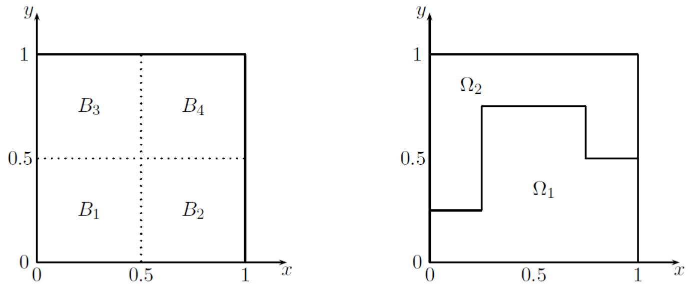

Fig. 1. Left: blocks, where $\kappa(\mu)$ is constant in space, right: domain decomposition of $\Omega = (0,1)^2$.

constant and depends on three parameters. Figure 1 shows the blocks, where $\kappa(\mu)$ is constant and the domain decomposition. This model leads to a weak form with

$$a(v, w; \mu) = \int_{\Omega} \kappa(\mu) \nabla v \cdot \nabla w \, dx, \quad v, w \in X, \mu \in \mathcal{P}.$$

The source consists of two exponential bubbles, with peaks in Ω_1 and Ω_2 and a fourth parameter as a weight between them:

$$f(v; \mu) = \int_{\Omega} h(\mu) v \, dx, \quad v \in X, \mu \in \mathcal{P},$$
$$h(x; \mu) = 80\mu_4 \exp\left(-20|x - z_1|^2\right) + 80(1 - \mu_4) \exp\left(-20|x - z_2|^2\right),$$

for $x \in \Omega$, $\mu \in \mathcal{P}$ with $z_1 = (0.5, 0.5)^T$ and $z_2 = (0.875, 0.875)^T$. So the parameter vector is 4-dimensional; $\mathcal{P} \subset \mathbb{R}^4$. The linear output is defined as the mean value of $u(\mu)$ on $\Omega_s = [0, 0.25] \times [0.75, 1]$:

$$s(\mu) = l(u(\mu)) = \frac{1}{|\Omega_s|} \int_{\Omega_s} u(\mu) \, dx, \quad \mu \in \mathcal{P}.$$

The left-hand side of Figure 2 shows values of $s(\mu)$ for 100 randomly generated parameters. Our basis generation procedure yields bases of different sizes $N = N_1 + N_2$ and $M = M_1 + M_2$ for the primal and the dual approximation space. We define the error $e_s^{n_u,n_\psi}(\mu) = |s(\mu) - s_N^{n_u,n_\psi}(\mu)|$ and the effectivity $\eta_s^{n_u,n_\psi}(\mu) = \Delta_s^{n_u,n_\psi}(\mu)/e_s^{n_u,n_\psi}(\mu)$, where in the following $n_u = n_{u,\mathrm{acc}}(\mu)$ and $n_\psi = n_{\psi,\mathrm{acc}}(\mu)$ for the respective parameter. The right-hand side of Figure 2 shows that we obtain fairly good approximations and that the estimate is clearly related to the error. The effectivity is at the range of 10^2. Exemplary values of the effectivity are shown in Table 1.

To conclude, to primal-dual framework has been successfully transferred to the Dirichlet-Neumann RB method. The introduction of smoothed solutions in the output approximation allows a-posteriori error estimation in a straight-forward manner. The results meet the expectations to the method.

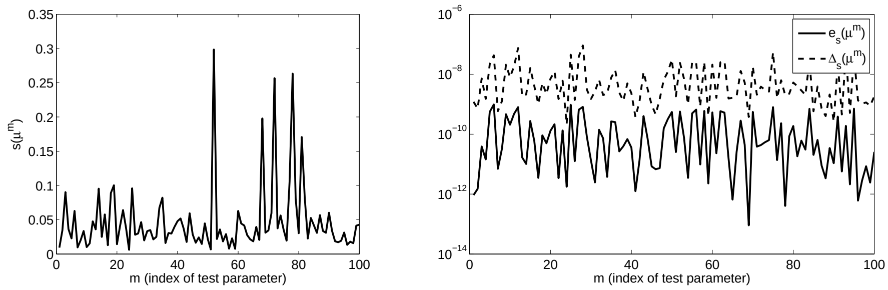

Fig. 2. Left: output values $s(\mu)$ on a parameter set of 100 randomly generated parameters. Right: investigation of the output error and corresponding estimate on the same parameter set with RB spaces of dimensions $N = 80$, $M = 28$.

Table 1. Output error $e_s^{n_u,n_\psi}(\mu)$, estimate $\Delta_s^{n_u,n_\psi}(\mu)$ and effectivity $\eta_s^{n_u,n_\psi}(\mu)$ for one randomly generated parameter and different bases sizes.

Bases sizes (N, M)	Output error	Estimate	Effectivity
$(61, 13)$	$7.65 \cdot 10^{-7}$	$7.23 \cdot 10^{-5}$	94.55
$(80, 28)$	$4.31 \cdot 10^{-11}$	$3.84 \cdot 10^{-9}$	89.18
$(85, 40)$	$6.94 \cdot 10^{-15}$	$2.78 \cdot 10^{-13}$	40.06

References

1. I. Maier and B. Haasdonk, *An iterative domain decomposition procedure for the reduced basis method*, submitted to APNUM 2012, revised 2013.
2. Y. Maday and E. Rønquist, *A reduced-basis element method*, Journal of Scientific Computing **17** (2002), 447–459.
3. L. Iapichino, *Reduced Basis Methods for the solution of Parametrized PDEs in Repetitive and Complex Networks with Application to CFD*, PhD thesis, EPFL (2012).
4. D. B. P. Huynh and D. Knezevic and A. T. Patera, *A static condensation reduced basis element method: Approximation and a posteriori error estimation*, Mathematical Modelling and Numerical Analysis **47(1)** (2013), 213–251.
5. C. Prud'homme and D. V. Rovas and K. Veroy and L. Machiels and Y. Maday and A. T. Patera and G. Turinici, *Reliable real-time solution of parametrized partial differential equations: Reduced-basis output bound methods*, Journal of Fluids Engineering **124** (2002), 70–80.
6. D. Rovas, *Reduced Basis Output Bound Methods for parametrized PDE's*, PhD thesis, MIT (2003).