Exercise 1: Parameter calculation

a) The hydraulic conductivities of two homogeneous porous media (sand, limestone) are

\[ K_{f, \text{sand}} = 4 \times 10^{-6} \, \text{m/s}, \quad K_{f, \text{lime}} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \times 10^{-9} \, \text{m/s}. \]

Which material is isotropic and which anisotropic? Why?

b) Consider single-phase (water) flow through porous media. The dynamic viscosity of water at 20°C is equal \( \mu = 0.001 \, \text{kg/m/s} \), the density of water is \( \rho = 1000 \, \text{kg/m}^3 \).

Compute the intrinsic permeabilities \( K_{\text{sand}} \) and \( K_{\text{lime}} \) for the materials from (a). What are the units for the intrinsic permeability?

Exercise 2: Capillary pressure, relative permeability

Consider a porous medium filled by two fluids. The wetting phase saturation is \( S_w = 0.55 \), and the residual water saturation is \( S_{wr} = 0.1 \).

a) Compute the capillary pressure \( p_c \) using the Brooks–Corey relationship for the entry pressure \( p_d = 2 \times 10^5 \, \text{Pa} \) and parameter \( \lambda = 2 \).

b) Compute the relative permeabilities for the wetting and non-wetting phases using the Brooks–Corey relationship with parameter \( \lambda = 2 \).

c) Which alternative relationship can be used for computing \( p_c \) and \( k_{ra}, \alpha \in \{w, n\} \)? What parameters are required for such calculations?

Exercise 3: Homogenization

Homogenization of the problem

\[ -\nabla \cdot \left( a^{\varepsilon}(x) \nabla u^{\varepsilon}(x) \right) = f(x), \quad x \in \Omega, \]
\[ u^{\varepsilon}(x) = 0, \quad x \in \partial \Omega \] \hspace{1cm} (1)

is given by

\[ -\nabla \cdot \left( A \nabla u(x) \right) = f(x), \quad x \in \Omega, \]
\[ u(x) = 0, \quad x \in \partial \Omega, \] \hspace{1cm} (2)
where the effective coefficients $\mathbf{A} = \left( a_{ij} \right)_{i,j=1,2}$ are computed as

$$a_{ij} = \int_{Y=[0,1] \times [0,1]} a(y) \left( \delta_{ij} + \partial_y w_j(y) \right) dy.$$

The functions $w_i$ are the solutions of the following BVP

$$-\nabla_y \cdot \left( a(y) \left( \nabla_y w_j(y) + e_j \right) \right) = 0, \quad y \in Y, \quad w_j(y + e_i) = w_j(y), \quad y \in \partial Y,$$

where $i = 1, 2$ and $j = 1, 2$.

a) Is $\mathbf{A}$ isotropic? Is $\mathbf{A}$ diagonal?

b) Is $\mathbf{A}$ symmetric? If yes, prove that, otherwise find an example.

c) Prove that $\mathbf{A}$ is positive definite.

*Use the lecture material concerning the cell problems.*